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Abstract

We propose a new multivariate DCC-GARCH model that extends existing approaches by admitting multivariate thresholds in conditional volatilities and conditional correlations. Model estimation is numerically feasible in large dimensions and positive semi-definiteness of conditional covariance matrices is naturally ensured by the pure model structure. Conditional thresholds in volatilities and correlations are estimated from the data, together with all other model parameters. We study the performance of our approach in some Monte Carlo simulations, where it is shown that the model is able to fit correctly a GARCH-type dynamics and a complex threshold structure in conditional volatilities and correlations of simulated data. In a real data application to international equity markets, we observe estimated conditional volatilities that are strongly influenced by GARCH-type and multivariate threshold effects. Conditional correlations, instead, are determined by simple threshold structures where no GARCH-type effect could be identified.

Keywords

Multivariate GARCH models; Dynamic conditional correlations; Tree-structured GARCH models; Model con¯dence set approach.

JEL Classification

C12, C13, C51, C53, C61.

1 Introduction

In this paper, we present a new DCC-GARCH model that extends existing approaches by admitting multivariate thresholds in conditional volatilities and correlations of multivariate time series. Such an extension, allows us to account for rich asymmetric effects and dependencies of conditional volatilities and correlations, as they are often encountered - for instance - in financial real data applications. Model estimation is numerically feasible in large dimensions. Moreover, positive semi-definiteness of conditional covariance matrices is ensured in a natural way by the structure of the model. Finally, thresholds in volatilities and correlations of our model are not fixed ex ante, but they are estimated from the data, together with all other parameters in the model.

To define such multivariate thresholds in the model, we extend the basic tree-structured state space partition in Audrino and Bühlmann (2001) to a setting with multivariate thresholds in volatilities and correlations. We present a feasible way of dealing with such rich threshold structures in conditional volatilities and correlations. To this end, we apply a binary tree structured threshold construction where each terminal node in the tree defines a local GARCH-type dynamics for volatilities and/or correlations over a partition cell of the whole (multivariate) state space. A simple two-step procedure is then proposed to estimate the joint local tree-structured GARCH-type dynamics for conditional volatilities and correlations, as well as the number and the structure of the underlying multivariate thresholds. The estimated optimal tree structure in volatilities and correlations is identified by solving a high dimensional model selection problem based on the Akaike Information Criterion[1] (AIC).

We study the performance of our general tree-structured DCC-GARCH model in some Monte Carlo simulations and focus on several multivariate volatility and correlation dynamics, including Engle (2002) pure DCC-setting, Audrino and Trojani (2003) CCC-setting with tree-structured volatility thresholds, Pelletier (2002) DCC-setting with switching regimes in dynamic correlations and, finally, a general multivariate tree-structured GARCH-DCC-setting including thresholds in conditional volatilities and/or correlations. Our simulations show that a tree-structured GARCH-DCC-based estimation procedure is able to identify correctly different complex multivariate GARCH-dynamics and threshold structures in conditional volatilities and correlations. The performance of the competing approaches, instead, is negatively affected under simple forms of a misspecification in conditional volatilities or correlations. For instance, Engle (2002) GARCH-DCC-model is shown to have a quite low prediction power when multivariate threshold structures in volatilities or correlations are present. Similarly, regime switching models for conditional correlations yield a moderate forecasting power when no changes in regime are present. Our tree-structured DCC-GARCH approach, instead, can account for the potential joint presence of GARCH effects and multivariate thresholds in conditional correlations and volatilities. Therefore, it offers a useful empirical strategy to identify and to disentangle such structures in real data.

We investigate the empirical performance of our model in a nine dimensional real data application to the estimation of the conditional volatility and correlation dynamics for the returns of international equity markets. We compare the performance of our methodology with the one of Engle (2002) GARCH-DCC-model, Ledoit et al. (2003) flexible multivariate GARCH model and Pelletier (2002) DCC-switching regime model. The first two models do not include multiple regimes or thresholds in volatilities or correlations. The third model includes a very simple regime structure in dynamic conditional correlations. Our setting, instead, encompasses GARCH-type dynamics and multivariate threshold structures in conditional volatilities and correlations. Moreover, as mentioned, potentially existing threshold structures are not fixed from the beginning. Therefore, our empirical study of international equity markets produces empirical evidence about the relative importance of GARCH-type effects and multivariate thresholds for volatilities and correlations in the dynamics of international equity returns.

Based on several in-sample and out-of-sample performance measures we find that our model outperforms the relevant competitors across several dimensions. In particular, we observe estimated conditional volatilities of equity index returns that are strongly influenced by both GARCH-type and multivariate threshold effects. Conditional correlations, instead, are strongly determined by some simple piecewise constant multivariate threshold structures, where no GARCH-type effect could be identified. Conditional volatility thresholds are mainly determined, as in Audrino and Trojani (2003), by the interaction of US market index (S&P500) returns with the other equity return series analyzed in this paper. Somehow surprisingly, multivariate conditional correlation thresholds are not affected by US index returns. Instead, they are only influenced by the CAC40-index, the DAX30-index and the NIKKEI-index returns. Such findings are consistent with the large empirical finance literature highlighting volatility and correlation spillovers between financial markets. For instance, the strong effects of US-index returns in our estimated volatility thresholds is supported by several multi-country studies highlighting strongly correlated international equity markets and volatility spillovers between the world's major trading areas.[2] The repeated empirical observation that international investors often overreact to US equity index returns[3] further motivates our findings about conditional volatility structures. Interestingly, however, our results also suggest that while US-index news are important to explain the asymmetric dependencies of conditional volatilities in global equity markets, their role in determining potential asymmetric correlation patterns might be not as strong.

In Section 2 we present our tree-structured threshold GARCH-DCC model and the estimation procedure that can be applied to estimate it. Section 3 presents our Monte Carlo simulations and the empirical findings from our application to the estimation of conditional volatility and correlation dynamics for international equity markets. Section 4 concludes and gives suggestions for future developments.

2 The model

In this section, we first introduce our multivariate tree structured models for conditional volatilities and correlations. In a second step, the estimation procedure needed to estimate such models is described.

2.1 Starting point

Let $(\mathbf{X}_t)_{t\in\mathbb{Z}}$ be a stochastic process with values in $\mathbb{R}^d$. For exposition purposes, we start from a general (nonparametric) model of the form

$$\mathbf{X}_t = \Sigma_t Z_t, \tag{2.1}$$

where $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of iid zero mean innovations with values in $\mathbb{R}^d$ and the $d \times d$ matrix Σ_t is such that the conditional covariance matrix V_t can be factorized as

$$V_t := Cov_{t-1}(\mathbf{X}_t) = \Sigma_t \Sigma_t' = D_t R_t D_t. \tag{2.2}$$

Matrix

$$D_t = diag[\sigma_{t,1}, .., \sigma_{t,d}]$$

is a diagonal matrix of conditional volatilities $\sigma_{t,i}$ of the coordinates $X_{t,1}, .., X_{t,d}$ in vector $\mathbf{X}_t$, whereas R_t is a matrix of conditional correlations between such components:

$$R_t := corr_{t-1}(\mathbf{X}_t) := [\rho_{t,ij}]_{1\leq i,j\leq d}\ .$$

To simplify the notation, conditional means of $\mathbf{X}_t$ have been set to zero in (2.1). In the sequel, we consider models where D_t and R_t can be written as

$$D_t = g(\mathbf{X}_{t-1}, D_{t-1}) \ , \ R_t = f(\mathbf{X}_{t-1}, \epsilon_{t-1}, R_{t-1}),$$

where $\epsilon_t = D_t^{-1}\mathbf{X}_t$ and for some suitable functions f and g. Conditional volatilities are functions of lagged observations $\mathbf{X}_{t-1}$ and lagged conditional volatilities D_{t-1}. Conditional correlations are functions of lagged observations $\mathbf{X}_{t-1}$, lagged shocks ϵ_{t-1} and lagged conditional correlations R_{t-1}. The dependence of D_t on D_{t-1} and $\mathbf{X}_{t-1}$ can take into account a (possibly nonlinear) autoregressive structure of volatilities and an asymmetric dependence of volatilities on lagged process observations $\mathbf{X}_{t-1}$. Similarly, the dependence of R_t on R_{t-1}, $\mathbf{X}_{t-1}$ and ϵ_{t-1} can allow for a broad variety of asymmetric conditional correlation patterns.

We propose a parametric tree structured DCC-GARCH model for (2.1), which admits a high flexibility in the functional form of g and f. In particular, multivariate GARCH-type processes with dynamic conditional correlations, as in Engle (2002), are special cases of our model. Similarly, several types of CCC-GARCH-type models with multivariate thresholds in conditional volatilities, as for instance in Audrino and Bühlmann (2001) and Audrino and Trojani (2003), are special cases of our setting. Our model is parsimonious enough to be statistically and computationally manageable, when fitted using amounts of data that are typically available in applications. We accomplish this by means of two modelling steps. First, we partition the domains of f and g in a finite number of cells that partition the (multivariate) state space of $(\mathbf{X}_{t-1}, D_{t-1})$ and $(\mathbf{X}_{t-1}, \epsilon_{t-1}, R_{t-1})$, respectively. Second, for any given partition cell we specify a cell dependent GARCH structure for conditional variances D_t^2 and a cell dependent Engle (2002)-type DCC structure for conditional correlations R_t. In this way, our model allows for general threshold structures and asymmetric dependencies in both conditional volatilities D_t and conditional correlations R_t. Such threshold structures imply rich multivariate regime dynamics for the corresponding conditional covariance matrices of $\mathbf{X}_t$. Despite the existence of multivariate regimes in volatilities and correlations, the particular way of defining conditional variances and correlations in our model still allows us to estimate it by a two-step procedure where volatility dynamics are fitted in a first step and correlation dynamics are fitted in a second separate step. Such a property of our tree structured threshold model makes it applicable also for quite high dimensional settings.

2.2 Tree structured DCC-GARCH threshold models

Tree structured DCC-GARCH models parameterize the conditional volatilities $D_t = D_t(\theta_1)$ and correlations $R_t = R_t(\theta_2)$ in model (2.1) by means of some parametric threshold functions and a parameter vector $\theta = (\theta_1, \theta_2)'$:

$$\mathbf{X}_t = \Sigma_t(\theta)\epsilon_t \ , \tag{2.3}$$

where

$$D_t(\theta_1) = g_{\theta_1}(\mathbf{X}_{t-1}, D_{t-1}(\theta_1)) \quad , \quad R_t(\theta_2) = f_{\theta_2}(\epsilon_{t-1}, \mathbf{X}_{t-1}, R_{t-1}(\theta_2)) \ , \tag{2.4}$$

for some parametric functional forms g_{θ_1} and f_{θ_2}. In our model, we specify $D_t(\theta_1)$ as a threshold GARCH(1,1)-type function, where each diagonal element of $D_t(\theta_1)$ is modeled according to a univariate tree-structured threshold GARCH(1,1)-model, as in Audrino and Bühlmann (2001) and Audrino and Trojani (2003). The conditional correlation function $R_t(\theta_2)$ is modeled according to a threshold DCC-type function described more precisely below.

2.2.1 Tree structured model for $D_t(\theta_1)$

Function

$$D_t(\theta_1) = diag[\sigma_{t,1}(\theta_{1,1}), .., \sigma_{t,d}(\theta_{1,d})]$$

is defined by d univariate tree structured GARCH(1,1) dynamics for each single conditional variance component $\sigma^2_{t,j}(\theta_{1,j})$, where $j = 1, .., d$. More precisely, let $X_{t,j}$ be the $j-$th component of $\mathbf{X}_t$. For illustration purposes, we assume that threshold structures in the $X_{t,j}$ volatility dynamics only depend on components $X_{t-1,1}$ and $X_{t-1,j}$ in $\mathbf{X}_{t-1}$. In our application to international equity markets, component $X_{t,1}$ will be the return on the US equity index at time t.[4]

To define threshold function $\sigma^2_{t,j}(\theta_{1,j})$ in our model, let $\mathcal{P}_j = \{\mathcal{R}_{1,j}, .., \mathcal{R}_{k_j,j}\}$ be a partition of the state space $G := \mathbb{R}^2 \times \mathbb{R}^+$ of $(X_{1,t-1}, X_{j,t-1}, \sigma^2_{t-1,j}(\theta_{1,j}))$. Given a partition cell $\mathcal{R}_{i,j}$, the local conditional dynamics of $X_{t,j}$ on $\mathcal{R}_{i,j}$ are defined by a GARCH(1,1) model. Therefore, threshold function $\sigma^2_{t,j}(\theta_{1,j})$ is of the form

$$\sigma^2_{t,j}(\theta_{1,j}) = g^{\mathcal{P}_j}_{\theta_{1,j}}(X_{1,t-1}, X_{j,t-1}, \sigma^2_{t-1,j}(\theta_{1,j})), \tag{2.5}$$

where

$$g^{\mathcal{P}_j}_{\theta_{1,j}}(X_1, X_j, \sigma^2_j) = \sum_{i=1}^{k_j}(\alpha_{ij} + \beta_{ij}X^2_j + \gamma_{ij}\sigma^2_j)I_{[(X_1,X_j,\sigma^2_j)\in\mathcal{R}_{i,j}]},$$

parameter $\theta_{1,j}$ is given by

$$\theta_{1,j} = \{\alpha_{ij}, \beta_{ij}, \gamma_{ij} \ ; \ i = 1, .., k_j\}$$

and $I_{[\cdot]}$ is an indicator function.

To completely specify function $g_{\theta_{1,j}}^{\mathcal{P}_j}$, we have to define the class of partitions $\mathcal{P}^j$ which are admissible in our tree-structured model. Essentially, the only restriction is that $\mathcal{P}^j$ has to be composed by *rectangular* partition cells $\mathcal{R}_{ij}$, $i = 1, .., k_j$, delimited by a set of multivariate thresholds for $(X_{1,t-1}, X_{j,t-1}, \sigma_{t-1,j}^2(\theta_{1,j}))$. Such rectangular partition cells are obtained by means of a binary tree $\mathcal{T}^j$ where every terminal node represents a particular cell $\mathcal{R}_{ij}$. Details on the construction and the interpretation of such binary trees in relation to our application to international equity markets are provided in Audrino and Trojani (2003).

For each component $X_{t,j}$, estimation of $g_{\theta_{1,j}}^{\mathcal{P}_j}$ is achieved by a high dimensional model selection problem that determines the optimal number and the structure of the relevant thresholds (and hence the partition cells) in $\mathcal{P}_j$. Details on such an estimation procedure for univariate tree structured GARCH(1,1) models are provided in Audrino and Bühlmann (2001) and Audrino and Trojani (2003), Section 2.3.

2.2.2 Tree structured model for $R_t(\theta_2)$

Let

$$\epsilon_t = D_t(\theta_1)^{-1}\mathbf{X}_t,$$

so that

$$R_t = Corr_{t-1}(\mathbf{X}_t) = Cov_{t-1}(\epsilon_t).$$

We model R_t by means of a tree structured approach where conditional correlations satisfy a Engle (2002)-type local DCC-model across several multivariate regimes. To this end, we define

$$R_t = diag[Q_t]^{-1/2} Q_t diag[Q_t]^{-1/2}$$

and model the dynamics of matrix

$$Q_t = [q_{t,ij}]_{1 \le i,j \le d}$$

by a parametric tree structured DCC-model $Q_t(\theta_2)$. Hence, we introduce multivariate thresholds in R_t by allowing for multivariate thresholds in the conditional dynamics of Q_t.

To maintain tractability of the model, we assume that thresholds in the Q_t dynamics depend on ϵ_{t-1} via the average

$$\overline{\rho}_{t-1} = \frac{1}{d(d-1)} \sum_{u \neq v} \epsilon_{t-1,u}\epsilon_{t-1,v}$$

of the cross products of the component of ϵ_{t-1}. Intuitively, this choice allows us to account for asymmetric effects in conditional correlations as a function of both particular lagged process realizations $\mathbf{X}_{t-1}$ and specific movements in average lagged conditional correlations shocks $\overline{\rho}_{t-1}$.

To define the parametric threshold function $Q_t(\theta_2)$ in our model, let $\mathcal{P} = \{\widetilde{\mathcal{R}}_1, .., \widetilde{\mathcal{R}}_w\}$ be a partition of the state space $\widetilde{G} := \mathbb{R}^{d+1}$ of $(\mathbf{X}_{t-1}, \overline{\rho}_{t-1})$. Given a partition cell $\widetilde{\mathcal{R}}_i$, the local conditional dynamics of ϵ_t on $\widetilde{\mathcal{R}}_i$ are defined by an Engle (2002)-type DCC model. Therefore, threshold function $Q_t(\theta_2)$ is of the form

$$Q_t(\theta_2) = f_{\theta_2}^{\mathcal{P}}(\mathbf{X}_{t-1}, \epsilon_{t-1}, Q_{t-1}(\theta_2)), \tag{2.6}$$

where function $f_{\theta_2}^{\mathcal{P}}$ is given by

$$f_{\theta_2}^{\mathcal{P}}(\mathbf{X}, \epsilon, Q) = \sum_{i=1}^{w} c_i[(1 - \phi_i - \lambda_i)\overline{Q} + \phi_i \epsilon\epsilon' + \lambda_i Q] \cdot I_{[(\mathbf{X}, \overline{\rho}) \in \widetilde{\mathcal{R}}_i]}, \tag{2.7}$$

with parameters $c_j \in (0, 1]$, $\phi_j, \lambda_j \geq 0$ such that $\phi_j + \lambda_j < 1$ for any $j = 1, .., w$, and $\overline{Q} \in \mathbb{R}^{2d}$. Parameter vector θ_2 is then given by

$$\theta_2 = \{c_i, \phi_i, \lambda_i, vech(\overline{Q}) \ ; \ i = 1, .., w\}.$$

Since for any $i = 1, .., d$ the local model for Q_t in (2.7) satisfies a Engle (2002) DCC-type dynamics, positive definiteness of the resulting threshold model $Q_t(\theta_2)$ (and hence also the threshold model for $R_t(\theta_2)$) is implied by the pure model structure under the above conditions on the model parameters.

By construction, when $\mathcal{P} = \{\widetilde{G}\}$ is a trivial partition and $c_i = 1$ for $i = 1, .., w$, we obtain Engle (2002) DCC-setting. Therefore, such a model is nested in our one. If $\phi_i = \lambda_i = 0$ for $i = 1, .., w$, then:

$$f_{\theta_2}^{\mathcal{P}}(\mathbf{X}, \epsilon, Q) = \sum_{i=1}^{w} c_i \overline{Q} \cdot I_{[(\mathbf{X}, \overline{\rho}) \in \widetilde{\mathcal{R}}_i]},$$

and we obtain a piecewise constant function $f_{\theta_2}^{\mathcal{P}}$ over the given partition $\mathcal{P}$. Such a type of piecewise constant function is the optimal one that has been estimated in our application to international equity markets in Section 3.2. More generally, when $\phi_i > 0$ or $\lambda_i > 0$ for some

$i = 1, .., w$ and $\mathcal{P}$ is not a trivial partition, we obtain a tree structured DCC dynamics satisfying locally Engle's DCC-model over the single partitioning cells $\widetilde{\mathcal{R}}_i$, $i = 1, .., k$.

As for the tree structured volatility models in the last section, in order to fully specify function $f_{\theta_2}^{\mathcal{P}}$ we have to define the class of partitions $\mathcal{P}$ which are admissible in our tree-structured model for conditional correlations. Again, the only restriction we put on $\mathcal{P}$ is that it is composed by *rectangular* partition cells $\widetilde{\mathcal{R}}_i$, $i = 1, .., w$. Consistently with our model (2.7), such partition cells are now delimited by a set of multivariate thresholds for $(\mathbf{X}_{t-1}, \overline{\rho}_{t-1})$. We make use of a binary tree $\mathcal{T}$ where every terminal node represents a cell $\widetilde{\mathcal{R}}_i$ in order to construct such rectangular partition cells. Estimation of $Q_t(\theta_2)$ can be then obtained with a high dimensional model selection problem that determines the optimal number and the structure of the relevant thresholds (and hence the partition cells) in $\mathcal{P}$.

Such a direct high dimensional model selection scheme to determine the threshold structure of the threshold function (2.7) is not computationally feasible, when applied directly to the multivariate time series $(\epsilon_t)_{t\in\mathbb{Z}}$. However, a natural way to reduce estimation complexity is to remark that the partition $\mathcal{P}$ in (2.7) is identical to the one implied by a corresponding tree structured univariate ARMA(1,1)-type model for the time series $(\overline{\rho}_t)_{t\in\mathbb{Z}}$. Indeed, since

$$E_{t-1}(\epsilon_t\epsilon_t') = Cov_{t-1}(\epsilon_t) = R_t = Q_t^{-1/2} Q_t Q_t^{-1/2},$$

it follows from (2.7):

$$\begin{aligned} E_{t-1}(\overline{\rho}_t) &= \sum_{u\neq v} q_{t,uv}/[d(d-1)] \\ &= \sum_{i=1}^{w} c_i[(1-\phi_i-\lambda_i)\overline{\overline{q}} + \phi_i\overline{\rho}_{t-1} + \lambda_i E_{t-2}(\overline{\rho}_{t-1})] \cdot I_{[(\mathbf{X}_{t-1},\overline{\rho}_{t-1})\in\widetilde{\mathcal{R}}_i]}, \end{aligned}$$

where $\overline{\overline{q}}$ is the average of the components of $\overline{Q}$ in (2.7) outside the main diagonal. Therefore, the tree structured model

$$\overline{\rho}_t = E_{t-1}(\overline{\rho}_t) + \eta_t, \tag{2.8}$$

where $(\eta_t)_{t\in\mathbb{Z}}$ is a martingale difference process, defines a univariate tree structured ARMA(1,1)-type process for $\overline{\rho}_t$ based on the same partition $\mathcal{P}$ as in (2.7).

Univariate model (2.8) can be used to estimate the threshold structure in (2.7). In particular, we can make use of (2.8) to test if $\mathcal{P} = \{\widetilde{G}\}$ is a trivial partition and $c_i = 1$ for $i = 1, .., w$, i.e. to test if Engle (2002) DCC-model holds. Similarly, we can test if $\phi_i = \lambda_i = 0$ for $i = 1, .., w$, i.e. if

$$f_{\theta_2}^{\mathcal{P}}(\mathbf{X}, \epsilon, Q) = \sum_{i=1}^{w} c_i \overline{Q} \cdot I_{[(\mathbf{X},\overline{\rho}) \in \widetilde{\mathcal{R}}_i]},$$

defines a piecewise constant function $f_{\theta_2}^{\mathcal{P}}$ over the given partition $\mathcal{P}$. Such a type of piecewise constant function is the optimal one that has been estimated in our application to international equity markets in Section 3.2.

Once partition $\mathcal{P}$ in (2.8) has been estimated, the model parameter θ_2 in (2.7) can be estimated using a multivariate conditional pseudo likelihood for ϵ_t where partition $\mathcal{P}$ in (2.7) is held fixed. The next section gives more details on the concrete estimation procedure of our tree structured DCC model.

2.3 Estimation of the tree structured DCC model

Estimation of our tree structured DCC model is achieved in two steps. In the first step, an estimate of matrix $D_t(\theta_1)$ is obtained by performing d estimations of the univariate tree structured conditional volatility dynamics $\sigma_{t,1}(\theta_{1,1}), .., \sigma_{t,d}(\theta_{1,d})$ implied by (2.5). The resulting estimate $\widehat{D}_t := D_t(\widehat{\theta}_1)$ is used to compute estimated scaled residuals

$$\widehat{\epsilon}_t := \widehat{D}_t^{-1} \mathbf{X}_t. \tag{2.9}$$

Scaled residual $\widehat{\epsilon}_t$ are used in the second step of our procedure to estimate the tree structured conditional correlation dynamics implied by (2.7).

2.3.1 Estimation of tree structured univariate GARCH-dynamics

Estimation of the d tree structured univariate volatility functions (2.5) is achieved by a high dimensional model selection problem, which determines the optimal structure of the relevant thresholds in any partition $\mathcal{P}_j$ of the univariate volatility dynamics (2.5), $j = 1, .., d$.

More specifically, in a first step a largest univariate tree structured GARCH model is estimated for any $j = 1, .., d$, given a fixed maximal number M_j of possible thresholds in (2.5). This first step delivers a maximal possible partition $\mathcal{P}_j^{max}$ of the relevant state space in the univariate volatility dynamics (2.5). Because of the tree structured construction of $\mathcal{P}_j^{max}$, this first step implies a maximal number $M + 1$ of conditional volatility regimes. A parsimonious specification of the maximal number M of thresholds ensures a statistically and computationally tractable model dimension. Moreover, it avoids (over) fitting a too flexible model dynamics, which would result in a poor out of sample forecasting power. In all our simulations and in our application to international equity markets, we fix the maximal number of candidate volatility thresholds for any univariate dynamics in our model at $M_j = 4$.

In a second step, a tree-structured model selection procedure for non nested models is applied, which selects the optimal subpartition $\mathcal{P}_j \subset \mathcal{P}_j^{max}$ out of the maximal one. Model selection is performed according to the AIC criterion implied by a conditionally gaussian log likelihood for any process coordinate $X_{t,j}$, $j = 1, .., d$. The resulting optimal tree-structured volatility model $g_{\theta_1,j}^{\mathcal{P}_j}$ minimizes the AIC criterion across all tree structured sub partitions of $\mathcal{P}_j^{max}$.

Details on the estimation of univariate tree structured GARCH(1,1) models, in relation to the modeling of international index dynamics, are provided in Audrino and Trojani (2003), Section 2.3.

2.3.2 Estimation of tree structured DCC-dynamics

We estimate the tree structured conditional correlation function (2.7) in two separate steps. In the first step, we estimate the optimal partition $\mathcal{P}$ using tree structured model (2.8) and scaled estimated residuals $\widehat{\epsilon}_t$. In the second step, we fix the partition $\widehat{\mathcal{P}}$ - say - estimated from model (2.8). We then estimate parameter θ_2 in (2.7) by a multivariate pseudo maximum likelihood estimator.

(i) Estimation of the univariate tree structured model (2.8). Let

$$\widehat{\overline{\rho}}_t = \sum_{u \neq v} \widehat{\epsilon}_{t-1,u} \widehat{\epsilon}_{t-1,v} / [d(d-1)]. \tag{2.10}$$

The following tree structured model for $\widehat{\overline{\rho}}_t$ is estimated (compare with equation (2.8)):

$$\widehat{\overline{\rho}}_t = E_{t-1}(\widehat{\overline{\rho}}_t) + \eta_t, \tag{2.11}$$

where $(\eta_t)_{t \in \mathbb{Z}}$ is a martingale difference sequence and

$$E_{t-1}(\widehat{\overline{\rho}}_t) = \sum_{i=1}^{w} c_i [(1 - \phi_i - \lambda_i)\overline{\overline{q}} + \phi_i \widehat{\overline{\rho}}_{t-1} + \lambda_i E_{t-2}(\widehat{\overline{\rho}}_{t-1})] I_{[(\mathbf{X}_{t-1}, \widehat{\overline{\rho}}_{t-1}) \in \widetilde{\mathcal{R}}_i]}.$$

This univariate tree structured model can be easily estimated with the general estimation procedures for tree structured models proposed in Audrino and Bühlmann (2001) and Audrino and Trojani (2003). We first estimate a largest univariate tree structured model, given a fixed maximal number M of possible thresholds in (2.8). In all our simulations and in our application to international equity markets, we fix the maximal number of candidate thresholds in model (2.8) at $M = 4$. A tree-structured model selection procedure for non nested models is then applied, which selects the optimal subpartition $\mathcal{P}$ out of the maximal one. Model selection is performed according to the AIC criterion implied by a conditionally gaussian pseudo log likelihood for $\widehat{\widehat{\rho}}_t$. See Audrino and Trojani (2003), Section 2.3, for details on such an estimation procedure.

At this stage of the estimation procedure, we also test with a bootstrap test (see for instance Efron and Tibshirani 1993) the hypothesis that (2.7) is piecewise constant:

$$\phi_i = \lambda_i = 0 \quad ; \quad i = 1, .., d, \tag{2.12}$$

and the hypothesis that (2.7) is a pure DCC-model:

$$\mathcal{P} = \{G\} \quad \text{and} \quad c_i = 1 \quad ; \quad i = 1, .., d. \tag{2.13}$$

If either hypothesis (2.12) or hypothesis (2.13) is not rejected, we impose it for the sequel of the estimation procedure of our tree structured DCC model. In such a case, the computational costs implied by the model estimation can be further reduced. In all our simulations in Section 3.1, such a testing procedure always identified correctly the underlying correlation dynamics, both when they satisfied hypothesis (2.12) or hypothesis (2.13). In our empirical application of Section 3.2, hypothesis (2.13) is rejected by the data, while hypothesis (2.12) is not.

(ii) Estimation of tree structured conditional correlation function (2.7). In the second step of our estimation procedure, we fix the partition $\widehat{\mathcal{P}}$ estimated for (2.8) in step (i) above and we estimate parameter θ_2 in (2.7) by a pseudo maximum likelihood estimator $\widehat{\theta}_2$ for θ_2, under a gaussian multivariate conditional pseudo likelihood for $\widehat{\epsilon}_t$. Moreover, if either hypothesis (2.12) or hypothesis (2.13) has not been rejected in step (i), we impose it in the definition of the relevant multivariate pseudo likelihood function when estimating θ_2. In our empirical application of Section 3.2, where hypothesis (2.12) is not rejected by the data, we therefore optimize only with respect to θ_2 a gaussian multivariate pseudo likelihood for $\widehat{\epsilon}_t$ such that function $f^{\mathcal{P}}_{\theta_2}$ in (2.7) is of the form

$$f^{\widehat{\mathcal{P}}}_{\theta_2}(\mathbf{X}, \widehat{\epsilon}, Q) = f^{\widehat{\mathcal{P}}}_{\theta_2}(\mathbf{X}, \widehat{\epsilon}) = \sum_{i=1}^{w} c_i \overline{Q} \cdot I_{[(\mathbf{X}, \widehat{\rho}) \in \widehat{\widehat{\mathcal{R}}}_i]},$$

where $\widehat{\mathcal{P}} := \{\widehat{\widehat{\mathcal{R}}}_1, .., \widehat{\widehat{\mathcal{R}}}_w\}$ is the estimated partition in step (i).

Together, steps (i) and (ii) above deliver the optimal estimated tree structured function $f^{\widehat{\mathcal{P}}}_{\widehat{\theta}_2}$ for the Q_t-dynamics defined by (2.7). Proofs of consistency of our tree structured model selection procedures for the case where the true model is in the class of tree-structured models (2.5) or (2.7) are very difficult to obtain. Analogously to CART, it is possible to prove theorems that study the behavior of the prevailing parameter estimators when growing the tree. However, such results do not imply model selection consistency. Furthermore, it is quite hard to believe that the "correct" generating process in our and similar real data examples is indeed exactly a tree-structured model (2.5) or (2.7) for volatilities and correlations, respectively. For this reason, it is more important to prove consistency of the estimates in a tree-structured model under a possible model misspecification, rather than showing consistency of the model selection strategy under the assumption of a correctly specified tree-structured model. Such consistency results can be found in Audrino and Bühlmann (2001). Based on such results, consistency of the two-step estimates $(\widehat{\theta}_1, \widehat{\theta}_2)$ in the tree structured DCC-GARCH model under a possible model misspecification can be derived in the standard way under mild regularity conditions; see, for instance, Newey and McFadden (1994) and Engle and Sheppard (2001).

3 Results

In this section, we test the in-sample and the out-of-sample performance of a tree structured DCC-GARCH(1,1) model in some Monte Carlo simulations and in an empirical application to the econometric analysis of international equity returns. In the Monte Carlo simulations, we primarily want to investigate the ability of our model to identify and to estimate correctly GARCH-type effects and/or multivariate thresholds in conditional volatilities and correlations. In the application to real data, we study the empirical evidence about the existence of GARCH-type effects and multivariate thresholds in conditional volatilities and correlations of international equity index returns.

In our Monte Carlo simulations, we compare the performance of our model with several competing ones. Some of those are nested in a general tree structured DCC-GARCH(1,1) model. More precisely, we compare performance with:

- A CCC-GARCH(1,1) model, as proposed by Bollerslev (1990); this model is nested in our one.
- A tree structured CCC-GARCH(1,1) model, as proposed by Audrino and Bühlmann (2001) and applied in Audrino and Trojani (2003) to the analysis of international equity markets; this model is nested in our one.
- A pure DCC-GARCH(1,1) model, as proposed by Engle (2002). This model is nested in our one.

- A DCC-GARCH(1,1) model with switching regimes in conditional correlations, as proposed in Pelletier (2002). This model is not formally nested in our one. It generates some simple switching regimes in conditional correlations in a way different from the one in our model.

In our empirical application we also include, in excess of the comparisons with the above models, one with the flexible multivariate GARCH model in Ledoit et al. (2003). That model does not include thresholds or regimes in conditional volatilities or correlations. However, it allows for a more general conditional correlations dynamics than Engle (2002) DCC model, and is therefore not nested in our model. Moreover, flexible multivariate GARCH models have been shown by Ledoit et al. (2003) to describe quite accurately the dynamics of international equity returns. Therefore, they are further natural competitors to our approach, especially in applications studying the multivariate dynamics of international equity markets.

In all our simulations and empirical studies, we work with a maximal number $M + 1$ of partition cells in the trees for conditional volatilities and correlations, where $M = 4$. This implies a maximal number of 5 regimes for conditional volatilities and/or correlations in our estimated models. For any coordinate axis of the multivariate state space that has to be splitted to determine the thresholds, we search over grid points that are empirical $\alpha-$quantiles of the data along the relevant coordinate axis. We fix the empirical quantiles as $\alpha = i/\text{mesh}, i = 1, .., \text{mesh}-1$, where mesh determines the fineness of the grid on which we search for multivariate thresholds. In all our estimations we fixed mesh $= 8$.

3.1 Monte Carlo Simulations

We simulate several special cases of the general model (2.1) under different constraints on either individual (squared) volatility functions or the dynamic conditional correlation structures. Multivariate iid innovations Z_t in (2.1) are generated in all simulations according to a multivariate standard normal distribution.

We split our simulations in three distinct exercises that allow us to investigate to which extent our tree structured approach is able to identify correctly the presence of threshold structures (i) only in volatilities, (ii) only in conditional correlations and (iii) in volatilities and correlations. For simplicity and for brevity, we perform such exercises in the low dimensional case of a bivariate time series. A high dimensional simulation exercise in presented in Section 3.1.4, to test our approach also in such a more computationally involved context.

To quantify forecasting power and to compare statistical accuracy across models we adopt several measures of in-sample and out-of-sample performance. We also test formally for differences in forecasting power between models. Forecasting power and statistical adequacy are measured according to the model's ability to forecast individual (squared) conditional volatilities and correlations.

We first analyze the goodness of standardized residuals $\mathbf{Z}_t = \Sigma_t^{-1}\mathbf{X}_t$ in (2.1) for the model estimates arising under the different competing approaches. Such a comparison is based on the goodness of fit criteria proposed in Engle and Sheppard (2001). Under a correctly specified model (2.1), Z_t must have a constant conditional covariance matrix equal to the identity matrix. Moreover, the cross products $Z_t Z_t'$ must be uncorrelated over time. Therefore, it is natural to test whether:

1. standardized residuals Z_t estimated by the different approaches have an identity covariance matrix,

2. estimated cross products $Z_t Z_t'$ are uncorrelated over time.

To test the null hypothesis 1., we can study the percentage of standardized residuals with variance in a given confidence interval of one. To test the null hypothesis 2., we can study the percentage of rejections in a standard Ljung-Box test for the presence of excess serial correlation in some products of standardized residuals, up to the 15^{th} lag and at a confidence level of 5%.

To quantify in-sample and out-of-sample forecasting power, we compute several in-sample and out-of-sample goodness-of-fit statistics for individual (squared) volatilities, conditional correlations and conditional covariances. We focus on measures based on:

- a multivariate negative log-likelihood statistic (NL),

- a multivariate version of the classical mean absolute error statistic (MAE),

- a multivariate version of the root mean squared error statistic (RMSE),

- the average absolute empirical correlation statistic between simulated values and one-step ahead predicted values of the quantities of interest (R^2). Averages are computed over all process coordinates.

More specifically, the following performance measures based on the above statistics will be considered (IS denotes in-sample and OS denotes out-of-sample):

$$\text{IS-NL: } -\text{log-likelihood}(\mathbf{X}_1^n; \widehat{\phi})$$

$$\text{OS-NL: } -\text{log-likelihood}\Big(\mathbf{Y}_1^{n_{\text{out}}}; \widehat{\phi}\Big)$$

$$\text{IS-MAE: } \frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n}\sum_{t=1}^{n} \mid v_{t,ij} - \widehat{v}_{t,ij} \mid$$

$$\text{OS-MAE: } \frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n_{\text{out}}}\sum_{t=1}^{n_{\text{out}}} \mid v_{t,ij} - \widehat{v}_{t,ij}(\mathbf{Y}_1^{t-1}) \mid$$

$$\text{IS-RMSE: } \Big(\frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n}\sum_{t=1}^{n} \mid v_{t,ij} - \widehat{v}_{t,ij} \mid^2\Big)^{1/2}$$

$$\text{OS-RMSE: } \Big(\frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n_{\text{out}}}\sum_{t=1}^{n_{\text{out}}} \mid v_{t,ij} - \widehat{v}_{t,ij}(\mathbf{Y}_1^{t-1}) \mid^2\Big)^{1/2}$$

$$\text{IS-R}^2\text{: } \frac{1}{d^2}\sum_{i,j=1}^{d} \mid \text{Cor}\Big(v_{t,ij}, \widehat{v}_{t,ij}\Big) \mid$$

$$\text{OS-R}^2\text{: } \frac{1}{d^2}\sum_{i,j=1}^{d} \mid \text{Cor}\Big(v_{t,ij}, \widehat{v}_{t,ij}(\mathbf{Y}_1^{t-1})\Big) \mid,$$

where in the OS performance measures the expression $\widehat{v}_{t,ij}(\mathbf{Y}_1^{t-1})$ is the $ij-$th covariance prediction implied by our out-of-sample data $\mathbf{Y}_1^{n_{\text{out}}} = \mathbf{Y}_1, \ldots \mathbf{Y}_{n_{\text{out}}}$ at time t under the parameter estimates obtained from in-sample data $\mathbf{X}_1^n = \mathbf{X}_1, \ldots \mathbf{X}_n$. That is, in all our simulations $\mathbf{X}_1^n$ and $\mathbf{Y}_1^{n_{\text{out}}}$ are independent realizations of the same simulated process. For our comparisons, the most important measures are the out-of-sample ones. With the exception of the R^2-related measures, in all other cases a lower performance measure indicates a higher forecasting power.

3.1.1 First simulation exercise

We first study forecasting ability for conditional correlations under a very simple structure for conditional volatilities. More specifically, we study the ability of our approach to identify thresholds in conditional correlations, when no threshold in conditional volatilities is present. Similarly, we can investigate the impact of a threshold-induced misspecification of conditional correlations for the results obtained by approaches competing with our model, like for instance Engle's (2002) DCC-setting.

To this end, we simulate from the general model (2.4) a sequence of 2000 observations (i.e. $n = n_{\text{out}} = 1000$) of a bivariate return series $\mathbf{X}_t$ under a simple GARCH(1,1) dynamics for individual (squared) volatilities. Individual volatility dynamics for the two process coordinates $X_{t,1}$ and $X_{t,2}$ are supposed to be both of the form:

$$v_{t,ii} = \sigma_{t,i}^2 = 0.15 + 0.1X_{t-1,i}^2 + 0.8\sigma_{t-1,i}^2 \;\; i = 1, 2. \tag{3.1}$$

Conditional correlations are supposed to exibit a simple, piecewise constant, threshold structure given by:

$$\rho_{t,12} = \begin{cases} 0.9\ , & \text{if } X_{t-1,1} \leq -1.2 \text{ and } X_{t-1,2} \leq -0.8, \\ 0.6\ , & \text{if } X_{t-1,1} \leq -1.2 \text{ and } X_{t-1,2} > -0.8, \\ 0.7\ , & \text{if } X_{t-1,1} > -1.2. \end{cases} \tag{3.2}$$

Parameter values in (3.1) and (3.2) are selected in order to mimic time series properties of real-data log-returns. The threshold structure in (3.2) allows for dynamic conditional correlations that can take different regime-dependent, but constant, values. Correlation regimes in (3.2) are characterized by thresholds depending on the lagged observations $X_{t-1,1}$ and $X_{t-1,2}$. The largest conditional correlation arise for very low values of $X_{t-1,1}$ and $X_{t-1,2}$. Such a choice is consistent with the intuition that conditional correlations are larger when all markets have suffered from large losses. The data generating process defined by (3.1) and (3.2) is nested in our tree structured DCC-GARCH(1,1) model.

The optimal threshold structure estimated in our tree structured DCC-GARCH(1,1) model yields a correct identification of the underlying dynamics in conditional correlations and volatilities. Our estimation procedure estimated correctly a GARCH(1,1) volatility function (3.1) for $X_{t,1}$ and $X_{t,2}$ with parameter vectors $(0.154, 0.127, 0.739)'$ and $(0.160, 0.093, 0.749)'$ that are quite close to the actual one $(0.150, 0.100, 0.800)'$. Furthermore, the second step of our estimation procedure could identify correctly the piecewise constant structure of conditional correlations. The estimated threshold function for conditional correlations is:

$$\widehat{\rho}_{t,12} = \begin{cases} 0.763\ , & \text{if } X_{t-1,1} \leq -1.231 \text{ and } X_{t-1,2} \leq -0.309, \\ 0.705\ , & \text{if } X_{t-1,1} \leq -1.231 \text{ and } X_{t-1,2} > -0.309, \\ 0.713\ , & \text{if } X_{t-1,1} > -1.231. \end{cases}$$

Differences between regime-dependent values of the estimated correlation parameter are moderate. Nevertheless, the estimated correlation structure does not exhibit, correctly, GARCH-type effects in its dynamics.

Results of the goodness-of-fit tests and performance measures introduced in Section 3.1 for the different approaches compared in the paper are presented in Table 1.

TABLE 1 ABOUT HERE.

Consistently with the simple GARCH(1,1) dynamics (3.1) assumed in this first exercise, estimated volatility dynamics selected by all tree-structured models analyzed in the paper are of a GARCH(1,1) type. Therefore, for all model settings to be compared in the sequel estimated conditional volatility dynamics are identical in this first exercise. Differences in performance across models arise only because of a different estimated conditional correlation structure. From this viewpoint, it is not surprising that the test statistics of our two tests for standardized residuals in Panel A of Table 1 are identical for all estimated models. When looking at the in-sample and the out-of-sample performance measures in Panel B of Table 1 we observe that, as expected, a tree structured model in volatilities and correlations (denoted by TreeDCC-TreeGARCH(1,1)) yields the best performance according to all performance statistics, with the only exception of the (in-sample) IS-NL measure. The improvement achieved in forecasting covariance matrices is, however, small: relative differences in forecasting performance range from about 1% to about 3%, depending on the measure used. Improvements in forecasts of true conditional correlations are, however, larger. For instance, when computing the MAE and the RMSE statistics directly on estimated conditional correlations, the increase in forecasting power implied by a TreeDCC-TreeGARCH(1,1) ranges from about 12% to about 17%, depending on the measure used. Therefore, even if a correct estimation of conditional correlations dynamics improve correlations forecasts, the improvement for conditional covariances forecasts might be actually small. This finding is not caused by the low dimension of the simulation exercise in this section. It will arise also for the high dimensional simulation in Section 3.1.4 and the real data application in Section 3.2. Such an evidence suggests that for applications focusing on a good prediction of conditional covariances, an accurate volatility forecasting model is probably more important than an accurate one for conditional correlations.

3.1.2 Second simulation exercise

We now study the forecasting accuracy of our tree structured model when multivariate thresholds in conditional volatilities and correlations are present. Therefore, we can analyze whether our approach is able to identify correctly the different threshold structures. Moreover, we investigate the impact of a threshold-induced misspecification of both conditional volatilities and correlations for the results obtained by the approaches competing with our model.

To this end, we simulate again 2000 observations (i.e. $n = n_{\text{out}} = 1000$) of a bivariate return series from the general model (2.4). In contrast to the previous simulation we now adopt two different volatility structures for the two process coordinates $X_{t,1}$ and $X_{t,2}$. We simulate conditional variances of $X_{t,1}$ according to the simple GARCH(1,1) model (3.1). Conditional variances of $X_{t,2}$ are simulated according to a tree structured threshold GARCH model given by:

$$\sigma_{t,2}^2 = \begin{cases} 0.15 + 0.4X_{t-1,2}^2 \;, & \text{if } X_{t-1,2} \leq 0, \\ 0.2 + 0.2X_{t-1,2}^2 + 0.75\sigma_{t-1,2}^2 \;, & \text{if } X_{t-1,2} > 0 \text{ and } \sigma_{t-1,2}^2 \leq 0.5, \\ 0.8 + 0.6\sigma_{t-1,2}^2 \;, & \text{if } X_{t-1,2} > 0 \text{ and } \sigma_{t-1,2}^2 > 0.5. \end{cases} \tag{3.3}$$

The thresholds in (3.3) depend only on lagged values of $X_{t,2}$ and not on those of $X_{t,1}$. Such a choice is applied only for simplicity and is not restrictive for the type of threshold functions that can be estimated in applications by our approach. Dynamic conditional correlations have the threshold structure (3.2), as in the previous simulation experiment. The data generating process with volatility and correlation functions (3.1), (3.3), (3.2) is nested in our model. The conditional volatility and correlation functions estimated by our model in this second experiment are:

$$\hat{\sigma}_{t,1}^2 = 0.098 + 0.102X_{t-1,1}^2 + 0.779\hat{\sigma}_{t-1,1}^2,$$

$$\hat{\sigma}_{t,2}^2 = \begin{cases} 0.141 + 0.343X_{t-1,2}^2 + 0.066\hat{\sigma}_{t-1,2}^2 \;, & \text{if } X_{t-1,2} \leq 0.012, \\ 0.085 + 0.142X_{t-1,2}^2 + 0.870\hat{\sigma}_{t-1,2}^2 \;, & \text{if } X_{t-1,2} > 0.011 \text{ and } \sigma_{t-1,2}^2 \leq 0.508, \\ 0.585 + 0.223X_{t-1,2}^2 + 0.593\hat{\sigma}_{t-1,2}^2 \;, & \text{if } X_{t-1,2} > 0.011 \text{ and } \sigma_{t-1,2}^2 > 0.508 \end{cases}$$

and

$$\widehat{\rho}_{t,12} = \begin{cases} 0.765 \;, & \text{if } X_{t-1,1} \leq -0.575 \text{ and } X_{t-1,2} \leq -0.188, \\ 0.747 \;, & \text{if } X_{t-1,1} \leq -0.575 \text{ and } X_{t-1,2} > -0.188, \\ 0.707 \;, & \text{if } X_{t-1,1} > -0.575. \end{cases}$$

Comparing the resulting threshold estimates with (3.1), (3.3), (3.2), we remark that the structure of all threshold functions in volatilities and correlations has been correctly identified by our approach. Differences between regime-dependent values of the estimated correlation parameter are again smaller than the correct ones. However, the estimated correlation structure does again not exhibit, correctly, GARCH-type effects in its dynamics.

Results of the goodness-of-fit tests and performance measures introduced in Section 3.1 for the different approaches compared in the paper are presented in Table 2.

TABLE 2 ABOUT HERE.

Panel B of Table 2 shows a clearly higher forecasting power of our tree structured model for volatilities and correlations, relatively to the competing approaches and across the different performance measures, with increases in forecasting power ranging from about 1% to about 50%. The largest performance increases are obtained when comparing the performance of our model with a competing one having misspecified volatility structure. For example, the increase in performance relatively to Engle (2002) DCC model (denoted by DCC-GARCH) is about 50%. Instead, the increase relatively to a tree structured CCC model including thresholds only in volatilities (denoted by CCC-TreeGARCH) is only about 1%. Improvements in forecasts of true conditional correlations are more significant. For instance, when computing the MAE and the RMSE statistics directly on estimated conditional correlations, the increase in forecasting power implied by a TreeDCC-TreeGARCH(1,1) model relatively to a CCC-TreeGARCH(1,1) one ranges from about 1% to about 5%. These results suggest again that for conditional covariances prediction purposes an accurate identification of the volatilities dynamics is probably more important than a precise estimation of conditional correlations.

3.1.3 Third simulation exercise

We now test whether our tree structured threshold model in volatilities and correlations is able to identify correctly also conditional correlation functions not of the threshold type. To this end, we simulate once more 2000 observations (i.e. $n = n_{\text{out}} = 1000$) of a bivariate return series from the general model (2.4). We adopt the same volatility structures (3.1), (3.3) for the two process coordinates $X_{t,1}$ and $X_{t,2}$, as in Section 3.1.2. Conditional correlations, however, are generated according to Engle (2002) DCC(1,1)-GARCH(1,1) model; see also (2.7) and the following remarks. The relevant parameters are:

$$\mathcal{P} = \{\widetilde{G}\},\ c_1 = 1,\ \phi = 0.05,\ \lambda = 0.85 \text{ and } \overline{Q} = [\overline{q}_{ij}]_{i,j=1,2} = \begin{bmatrix} 1 & 0.2603 \\ 0.2603 & 1 \end{bmatrix} \tag{3.4}$$

The data generating process with volatility and correlation functions (3.1), (3.3), (3.4) is nested in our model.

The optimal volatility structures for individual (squared) volatility functions estimated by our model in this simulation are:

$$\hat{\sigma}^2_{t,1} = 0.073 + 0.124X^2_{t-1,1} + 0.801\hat{\sigma}^2_{t-1,1}$$

and

$$\hat{\sigma}^2_{t,2} = \begin{cases} 0.144 + 0.343X^2_{t-1,2} + 0.006\hat{\sigma}^2_{t-1,2}\ , & \text{if } X_{t-1,2} \leq -0.014, \\ 0.237 + 0.192X^2_{t-1,2} + 0.356\hat{\sigma}^2_{t-1,2}\ , & \text{if } X_{t-1,2} > -0.014 \text{ and } \sigma^2_{t-1,2} \leq 0.325, \\ 0.229 + 0.156X^2_{t-1,2} + 0.575\hat{\sigma}^2_{t-1,2}\ , & \text{if } X_{t-1,2} > -0.014 \text{ and } \sigma^2_{t-1,2} > 0.325. \end{cases}$$

The estimated structure for dynamic conditional correlations is, correctly, a DCC(1,1) model. The estimated parameters are: $\widehat{\phi} = 0.0271$, $\widehat{\lambda} = 0.9512$ and $\widehat{\overline{q}}_{12} = 0.2684$. Thus, our tree structured model for volatilities and correlation has again identified correctly the structure of conditional volatilities and correlations. Moreover, the estimated parameters in the conditional correlation dynamics were estimated quite accurately.

Results of the goodness-of-fit tests and performance measures introduced in Section 3.1 for the different approaches compared in the paper are presented in Table 3.

TABLE 3 ABOUT HERE.

Table 3 highlights that Regime-Switching-type dynamic correlation models (denoted by RS-DCC-GARCH) cannot estimate accurately a dynamic conditional covariance structure of the form implied by the simple DCC(1,1) setting in Engle (2002). In fact, in our simulation these models perform even worse than a multivariate GARCH model with constant conditional correlations (denoted by CCC-GARCH). Our tree structured model, instead, identifies correctly the dynamics of conditional correlations and yields forecasting power improvements relatively to all competitors. In the present experiment, improvements in forecasting power of conditional covariances are often larger than those achieved in the previous simulations. In particular, very large improvements are obtained according to the OS-R^2 statistic. Again, such improvements derive mainly from a correct estimation of the individual threshold volatility functions. For instance, this is the case for the performance increases obtained with our general tree structured model relatively to Engle (2002) DCC one.

3.1.4 High-dimensional simulation exercise

We conclude our simulation analysis by considering a high dimensional dynamic conditional volatilities and correlations setting. To this end, we simulate 1000 observations (i.e. $n = n_{\text{out}} =$

500) of a fifty-dimensional time series from the general model (2.4). 5 randomly selected components in the fifty-dimensional multivariate process are obtained consistently with a volatility dynamics defined by means of the threshold function (3.3). The further process components are obtained consistently with a simple GARCH(1,1)-type volatility dynamics, as defined in (3.1). Since 90% of the simulated coordinate processes are defined by means of standard GARCH-type volatilities, the complexity in the multivariate dynamics generated by pure volatility effects is actually small, also compared, e.g., with the volatility structures estimated in our empirical application of Section 3.2. Simulated dynamic conditional correlations are defined by a simple tree structured process with piecewise constant correlations. The relevant threshold structure and the corresponding regime-dependent (constant) correlation parameters are defined by means of the following partition cells $\widetilde{\mathcal{R}}_1, \widetilde{\mathcal{R}}_2, \widetilde{\mathcal{R}}_3$ of the multivariate fifty-dimensional state space of X_t (see also (2.7)):

$$\begin{aligned}
\widetilde{\mathcal{R}}_1 &= \{X_{t-1,1} \leq -1.2 \text{ and } X_{t-1,2} \leq -0.8\},\ c_1 = 0.85 \\
\widetilde{\mathcal{R}}_2 &= \{X_{t-1,1} \leq -1.2 \text{ and } X_{t-1,2} > -0.8\},\ c_2 = 0.7 \\
\widetilde{\mathcal{R}}_3 &= \{X_{t-1,1} > -1.2\},\ c_3 = 0.8. \qquad (3.5)
\end{aligned}$$

Therefore, we are assuming that the conditional correlations threshold function in the model depends only on the first two process coordinates. Such variables can be interpreted, for instance, as some leading indicators for the multivariate process dynamics. The smallest correlations (i.e. with $c_2 = 0.7$) in the model arise when $X_{t-1,1} \leq -1.2$ and $X_{t-1,2} > -0.8$.

Results of the goodness-of-fit tests and performance measures introduced in Section 3.1 for the different approaches compared in the paper are presented in Table 4.

TABLE 4 ABOUT HERE.

As partly expected, since the complexity in the multivariate dynamics generated by pure volatility effects is actually small, all models perform quite satisfactorily with respect to (i) the percentage of standardized residuals with conditional variance in a confidence interval of one (for all models such percentage is 100%) and (ii) the percentage of rejections in the Ljung-Box tests (for all models such percentage is about 5%); see Panel A of Table 4 for details.

The findings in Table 4, Panel B, are consistent with the Monte Carlo evidence obtained in our previous low-dimensional simulation Exercises 1-3. Again, we find that our tree-structured model for volatilities and correlations performs best, according to the different out-of-sample performance indicators in Table 4. The largest improvements in forecasting power are obtained relatively to models where the dynamics of conditional volatilities are actually misspecified: Performance increases up to 5% relatively to models with misspecified volatilities are obtained, depending on the performance measure used. The increase in performance relatively to competing models with correctly specified volatilities but misspecificied conditional correlations is moderate, and is always less than 1%, under the different indicators of forecasting performance. Compared to the previous simulations, the increases in performance due to a correct specification of the estimated volatility functions is smaller because now only 10% of the process coordinates have been generated with volatility functions of the threshold type.

3.2 A real data application

We consider a multivariate time series of daily fully hedged USD log-returns for nine equity indices: the US S&P500 Index, the French CAC40 Index, the German Deutsche Aktien Index (DAX30), the Italian BCI General Index, the Canadian Toronto SE35 Index , the UK FT-SE-A All-Shares (FTSE100) Index, the Japanese NIKKEI225 Average Index, the Swiss SMI Index and the Hang Seng Index. Data are for the sample period between January 1, 1998 and November 4, 2002, amounting to 1262 trading days. All data have been downloaded from *Datastream International*. We split our full sample in two sub-periods. The first one consists of $n = 781$ trading days from January 1, 1998 to December 29, 2000. Data from this sample are used for in-sample estimation and performance evaluation. The second subsample consists of the remaining $n_{\text{out}} = 481$ observations up to November 4, 2002 and is used for out-of-sample performance evaluation purposes.

In our empirical analysis, we compare the performance of a tree-structured model for volatilities and correlations with the following competing models for the multivariate volatility and correlation dynamics:

- A CCC-GARCH(1,1) model, as proposed by Bollerslev (1990).
- A tree structured CCC-GARCH(1,1) model, as proposed by Audrino and Bühlmann (2001) and applied in Audrino and Trojani (2003) to the analysis of international equity markets.
- A pure DCC-GARCH(1,1) model, as proposed by Engle (2002).

- A DCC-GARCH(1,1) model with switching regimes in conditional correlations, as proposed in Pelletier (2002).

- A flexible DCC-GARCH(1,1) model, as in Ledoit et al. (2003).

All estimated models also include a linear autoregressive conditional mean function, denoted by μ_t. For each process coordinate $X_{i,t}$, the conditional mean function $\mu_{i,t}$ of $X_{i,t}$ depends only on the first lag $X_{i,t-1}$ and the first lag $X_{1,t-1}$ of the US-index return series, as in Audrino and Trojani (2003).

We focus on differences of forecasting power deriving from the different conditional covariance matrix dynamics under the different settings. To estimate our model with thresholds in conditional volatilities and correlations, we proceed as follows. We first estimate the univariate conditional volatility dynamics of the single series by including as possible conditioning variables in the threshold definition the first lag of the series itself, the prevailing conditional volatility of the series and the first lag of the US index return. Such a threshold volatility structure has been shown to produce good empirical results in Audrino and Trojani (2003). In the second estimation step of our procedures, we estimate possible dynamic conditional correlation structures. More precisely, we search for multivariate threshold structures in conditional correlations by including as potential threshold variables the first lag of all components of our multivariate index return series and the first lag of average conditional correlations shocks, as described in detail in Section 2.2.2.

We measure goodness of fit of the competing models by means of the same type of statistical tests on multivariate standardized residuals introduced in Section 3.1. We measure the prediction power of the different models by computing similar performance statistics for conditional covariance prediction as those introduced in Section 3.1 for our simulation exercises. More precisely, we compute in-sample and out-of-sample negative log-likelihood statistics (IS- and OS-NL) identical to those in the simulations. Since the IS- and OS- MAE, RMSE and R^2 performance measures can not be calculated exactly on real data - because they involve the "true" underlying conditional covariance matrix, which is unknown - we compute the following empirical estimates (denoted by an extra "P") for such quantities

$$\text{IS-PMAE: } \frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n}\sum_{t=1}^{n} \mid \widehat{v}_{t,ij} - (X_{t,i} - \widehat{\mu}_{t,i})(X_{t,j} - \widehat{\mu}_{t,j}) \mid$$

$$\text{OS-PMAE: } \frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n_{\text{out}}}\sum_{t=1}^{n_{\text{out}}} \mid \widehat{v}_{t,ij} - (\widetilde{X}_{t,i} - \widehat{\mu}_{t,i})(\widetilde{X}_{t,j} - \widehat{\mu}_{t,j}) \mid$$

$$\text{IS-PRMSE: } \Big(\frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n}\sum_{t=1}^{n} \mid \widehat{v}_{t,ij} - (X_{t,i} - \widehat{\mu}_{t,i})(X_{t,j} - \widehat{\mu}_{t,j}) \mid^2 \Big)^{1/2}$$

$$\text{OS-PRMSE: } \Big(\frac{1}{d^2}\sum_{i,j=1}^{d}\frac{1}{n_{\text{out}}}\sum_{t=1}^{n_{\text{out}}} \mid \widehat{v}_{t,ij} - (\widetilde{X}_{t,i} - \widehat{\mu}_{t,i})(\widetilde{X}_{t,j} - \widehat{\mu}_{t,j}) \mid^2 \Big)^{1/2}$$

$$\text{IS-PR}^2\text{: } \frac{1}{d^2}\sum_{i,j=1}^{d} \mid \text{Cor}\Big(\widehat{v}_{t,ij}, (X_{t,i} - \widehat{\mu}_{t,i})(X_{t,j} - \widehat{\mu}_{t,j})\Big) \mid$$

$$\text{OS-PR}^2\text{: } \frac{1}{d^2}\sum_{i,j=1}^{d} \mid \text{Cor}\Big(\widehat{v}_{t,ij}, (\widetilde{X}_{t,i} - \widehat{\mu}_{t,i})(\widetilde{X}_{t,j} - \widehat{\mu}_{t,j})\Big) \mid,$$

where n is the sample size used for model estimation, $\widetilde{\mathbf{X}}_1^{n_{\text{out}}} := \widetilde{\mathbf{X}}_1, \ldots, \widetilde{\mathbf{X}}_n = \mathbf{X}_{n+1}, \ldots, \mathbf{X}_{n+n_{\text{out}}}$ are the test data of the sample used for out-of-sample performance evaluation and all estimated conditional mean and covariance functions $\hat{\mu}$, $\hat{v}$ are defined by means of the model parameter estimates implied by the in-sample observations $\mathbf{X}_1^n = \mathbf{X}_1, \ldots, \mathbf{X}_n$.

Since IS- and OS-P measures in our empirical application are only noisy estimates of the correct ones, small differences in underlying correct goodness of fit measures could be obscured by a low signal to noise ratio when they are estimated. It is well known that in real data the noise component may be often dominant. Thus, our IS- and OS-P measures can be expected to discriminate only between forecasting models having quite large differences in forecasting power.

3.2.1 Estimation results

We estimated a TreeDCC-TreeGARCH(1,1) model using our nine-dimensional time series of international equity index returns. The estimated individual conditional mean and volatility thresholds are the same as those estimated in Audrino and Trojani (2003) and highlight the predominant role of lagged US S&P500 index returns in determining conditional volatility thresholds. Our goal in this section it to focus on the threshold structures estimated for conditional correlations. Optimal correlations threshold functions and parameters estimated for our tree structured DCC model are summarized in Table 5.

TABLE 5 ABOUT HERE.

Table 5 highlights that the most important variables affecting multivariate thresholds in conditional correlations are the DAX30 and the NIKKEI225 lagged index returns. More importantly, the estimated local correlation dynamics across the different regimes never contain GARCH-type effects, as opposed to, for instance, Engle (2002) DCC model.[5] Conditional correlations in each regime are constant. The full estimated threshold structure of conditional correlations in our model is completely characterized by means of only three lagged equity index returns: the one of the DAX30, the NIKKEI225 and the CAC40 (in order of statistical significance). More precisely, we estimated four different conditional correlation regimes in international equity returns. One regime is determined by low lagged DAX30 returns. The second one is due to large lagged DAX30 returns and low lagged NIKKEI225 returns. The third one is associated with contemporarily large lagged DAX30 and NIKKEI225 returns, together with low lagged CAC40 returns. Finally, the last one is caused by large lagged returns of all indices in the estimated threshold function.[6]

Results of the goodness-of-fit tests introduced in section 3.1 on the multivariate standardized residuals from different model fits are summarized in Table 6, Panel A.

TABLE 6 ABOUT HERE.

In Table 6, Panel A all models perform satisfactorily with respect to the percentage of standardized residuals with conditional variance in a confidence interval of one. Our model including a threshold specification in individual volatilities and dynamic conditional correlations performs slightly better than (i) models with constant conditional correlations and/or (ii) models with simple GARCH(1,1) dynamics for individual volatilities, when performance is measured by the percentage of null hypothesis rejections in the corresponding Ljung-Box tests. Such a preliminary evidence in favor of the higher performance of our model is strengthened by the prediction performance results in the next sections.

3.2.2 Multivariate performance results

We now analyze the accuracy of the conditional covariance matrix estimates and predictions implied by the different models under scrutiny. Results of goodness-of-fit prediction measures defined in Section 3.2 are summarized in Table 6, Panel B.

In Panel B of Table 6, we observe differences in estimated IS- and OS- PMAE and PRMSE statistics across models, which are quantitatively smaller than those typically observed in the simulations. This finding is a consequence of the fact that such statistics are noisy approximations of the underlying correct ones, with a noise component that is quite substantial in our real data analysis. Despite the small differences between estimated forecasting measures, the general message deriving from our empirical analysis is quite consistent. Our tree structured model for conditional volatilities and correlations achieves the best prediction results across all in-sample and out-of-sample measures used. Hence, our model performs best from the perspective of conditional covariance matrix prediction purposes in our real data example. Comparing the relative performance improvements deriving from the inclusion of threshold functions in individual conditional volatilities, as opposed to conditional correlations, we observe the following. For the NL and - especially - the MAE statistics, including thresholds in conditional correlations seems to add more forecasting power than including thresholds only in volatilities (to this end, compare the results obtained for the CCC-GARCH, the CCC-TreeGARCH and the TreeDCC-TreeGARCH models in Table 6). For the other statistics, improvements are larger when model complexity is increased by including threshold functions in conditional volatilities. This last finding is consistent with those obtained in our simulations, where forecasting power improvements due to a better conditional volatility prediction were larger than those deriving from a better model for conditional correlations.

3.2.3 Tests for equal predictive ability and model confidence set construction

In this section, we test formally for differences in forecasting power of the competing models, in order to select, if possible, a best one that significantly dominates the others in our real data application. To this end, we apply the Model Confidence Set (MCS) method proposed by Hansen et al. (2003) to characterize a best GARCH-type model out of a set of competing ones.

Let us denote by $\widehat{D}_{t,wk}$ the differences of each term in the OS-PMSE statistic[7]:

$$\widehat{D}_{t,wk} = \widetilde{U}_{t;\text{model}_w} - \widetilde{U}_{t;\text{model}_k},\ t = 1, \ldots, n_{\text{out}},\ w, k = 1, \ldots, 9, w < k,$$

where

$$\sum_{t=1}^{n_{\text{out}}} \widetilde{U}_{t;\text{model}} = \text{OS-PMSE}.$$

Similarly to Audrino and Bühlmann (2003), consider also the sign of $\hat{D}_{t,wk}$:

$$\widehat{W}_{t,wk} = \begin{cases} -1\ , & \text{if } \widehat{D}_{t,wk} \leq 0 \\ 1, & \text{else} \end{cases}\ ,\ t = 1, \ldots, n_{\text{out}},\ w, k = 1, \ldots, 9, w < k.$$

Statistics based on time averages $\overline{D}_{wk}$ and $\overline{W}_{wk}$ of $\widehat{D}_{t,wk}$ and $\widehat{W}_{t,wk}$ allow us to investigate, whether there is a systematic difference in out of sample forecasting power between the different models. Tests based on $\widehat{D}_{t,wk}$ are t-type tests, while test based on $\widehat{D}_{t,wk}$ are sign-type tests.

The MCS is defined as the smallest set of models which, at a given confidence level α, cannot be significantly distinguished based on their forecasting power. The MCS is determined after sequentially trimming the set of candidate models, which in our application consists of the six multivariate GARCH specifications introduced above. At each step of such a trimming procedure, the null-hypothesis of equal predictive ability (EPA) $\mathcal{H}_0 : \mathbb{E}[D_{t,wk}] = 0, \ \forall w, k \in \mathcal{M}$ (respectively, $\mathcal{H}_0 : \mathbb{E}[W_{t,wk}] = 0$) is tested for the relevant set of models $\mathcal{M}$ at a confidence level α. In a first step, $\mathcal{M}$ consists of all models under investigation. If, in the first step, $\mathcal{H}_0$ is rejected, then the worst performing model according to the relevant criterion is eliminated. The test procedure is then repeated for the new set $\mathcal{M}$ of surviving models, and it is iterated until the first non rejection of the EPA hypothesis occurs. The set of resulting models is called the model confidence set $\widehat{\mathcal{M}}_\alpha$ at the given confidence level α. In our application we work with $\alpha = 0.05, 0.10$.

Our tests of EPA are based on the range statistic T_R and the less conservative semi-quadratic statistic T_{SQ}:

$$T_R = \max_{k,w \in \mathcal{M}} \frac{|\overline{D}_{kw}|}{\sqrt{\widehat{\mathrm{var}}(\overline{D}_{kw})}} \quad \text{and} \quad T_{SQ} = \sum_{k<w} \frac{\overline{D}_{kw}^2}{\widehat{\mathrm{var}}(\overline{D}_{kw})},$$

where the sum in T_{SQ} is taken over the models in $\mathcal{M}$, $\overline{D}_{kw} = n_{\mathrm{out}}^{-1} \sum_{t=1}^{n_{\mathrm{out}}} \widehat{D}_{t,kw}$, and $\widehat{\mathrm{var}}(\overline{D}_{kw})$ is an estimate of $\mathrm{var}(\overline{D}_{kw})$ obtained from a block-bootstrap of the series $\widehat{D}_{t,kw}, \ t = 1, \ldots, n_{\mathrm{out}}$. Using statistics T_R or T_{SQ}, we test the null hypothesis EPA at confidence level α for model set $\mathcal{M}$. If hypothesis EPA is rejected for model set $\mathcal{M}$, we compute a worst performing index, in order to trim the worst performing model from $\mathcal{M}$.

The worst performing index for Model_k is computed as the mean across models $w \neq k$ of statistic $\overline{D}_{wk}$. More specifically, it is defined as $\overline{D}_k / \sqrt{\mathrm{var}(\overline{D}_k)}$, where $\overline{D}_k = \mathrm{mean}_{w \neq k \in \mathcal{M}} \overline{D}_{kw}$. As above, our estimate of $\mathrm{var}(\overline{D}_k)$ is based on a block-bootstrap. The model with the highest worst performing index is finally trimmed from $\mathcal{M}$. Consistency of estimates of the (asymptotic) distributions of T_R and T_{SQ} can be proved under mild regularity conditions on the bootstrap. For details, see Hansen et al. (2003).

Results of our t-type and sign-type tests for the hypothesis of EPA, as well as resulting model confidence sets in our application are summarized in Tables 7 and 8.

TABLES 7 AND 8 ABOUT HERE.

Table 7 shows that at the 10% confidence level, only our tree-structured model for volatilities and correlations survives, after trimming the worst performing models with the t-type range statistic. When using the t-type semi-quadratic statistic, the only rejected models are the CCC-GARCH and the Flexible M-GARCH models.

In Table 8, results for the more robust sign-type statistics are even more clear. In this case, the estimated model confidence set at the 10% confidence level only contains our tree-structured model for volatilities and correlations, when using the range statistic. With the semi-quadratic statistics, the only surviving two models are of the threshold in volatility type.

These results further confirm the higher out of sample forecasting power of tree-structured threshold models in our application.

4 Conclusions

We proposed a new multivariate tree-structured DCC-GARCH model that extends existing approaches by admitting multivariate thresholds in conditional volatilities and conditional correlations. Such thresholds are estimated from the data. Moreover, model estimation is feasible in large dimensions and positive semi-definiteness of conditional covariance matrices is ensured by the pure model structure. We studied the performance of our model in some simulations and in an application to international equity markets.

A comparison with several competing multivariate GARCH models, including Bollerslev's CCC model, Engle's DCC model and Pelletier's regime-switching DCC model, highlighted that such models have difficulties in fitting adequately the data when multivariate volatility or correlation thresholds are present. Our model, instead, was able to fit correctly GARCH-type dynamics, as well as complex threshold structures in conditional volatilities and correlations of simulated data.

In our real data application we found that estimated conditional volatilities are strongly characterized by both GARCH-type and multivariate threshold effects. Conditional correlations, instead, are determined by simple threshold structures where no GARCH-type effect could be identified. Estimated threshold structures imply a better out-of-sample forecasting performance of our model relatively to all competitors and across several measures of forecasting performance. The additional covariance matrix forecasting power achieved was larger for models including threshold structures in volatilities and correlations, compared to models including thresholds only in conditional correlations.

Notes

[1]We use AIC because of its good overall in sample and out of sample performances. However, AIC could be replaced by any sensible model selection criterion, such as for example the Schwarz Bayesian Information Criterion (BIC).

[2]See for instance Engle et al. (1990), Hamao et al. (1990), King and Wadhwani (1990), Bae and Karolyi (1994), Kim and Rogers (1995), Koutmos and Booth (1995), Chiang (1998).

[3]See for instance Becker et al. (1995).

[4]From a general viewpoint, also more general threshold structures and multivariate regimes could be considered in our modeling approach.

[5]In particular, the null hypothesis (2.12) of the test for constant conditional correlations is not rejected by the data. Hence, we fixed $\lambda_k = \phi_k = 0,\ k = 1, \ldots, d$. In contrast, null hypothesis (2.13) is rejected by the data.

[6]Note that with the terminology “low” (“large”) we mean in fact returns that are below (above) the thresholds in the estimated threshold functions.

[7]Results of tests for equal predictive ability based on the other performance statistics in the paper are identical to those obtained when using statistic OS-PMSE. We therefore omit them.

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Panel A

Model	% variance of stand. res. in CI	% Ljung-Box rejected
CCC-GARCH	100 (2/2)	25 (1/4)
DCC-GARCH	100 (2/2)	25 (1/4)
RS-DCC-GARCH	100 (2/2)	25 (1/4)
CCC-TreeGARCH	100 (2/2)	25 (1/4)
TreeDCC-TreeGARCH	100 (2/2)	25 (1/4)

Panel B

Model	IS-				OS-			
	NL	MAE	RMSE	R^2	NL	MAE	RMSE	R^2
CCC-GARCH	2686.3	0.1092	0.1533	0.8662	2724.2	0.1131	0.1605	0.8824
DCC-GARCH	2683.9	0.1105	0.1546	0.8643	2725.4	0.1140	0.1608	0.8814
RS-DCC-GARCH	2685.8	0.1092	0.1533	0.8662	2723.5	0.1131	0.1605	0.8824
CCC-TreeGARCH	2686.3	0.1092	0.1533	0.8662	2724.2	0.1131	0.1605	0.8824
TreeDCC-TreeGARCH	2685.9	0.1069	0.1493	0.8756	2722.7	0.1112	0.1563	0.8903

Table 1: Goodness-of-fit results from different model fits for a two-dimensional data example simulated from the general model (2.4) with threshold volatility and conditional correlation functions given in (3.1) and (3.2). Panel A: Tests on multivariate standardized residuals. Percentages of in-sample multivariate standardized residuals having variance in a confidence interval of one and percentages of rejected classical Ljung-Box tests investigating whether there is excess serial correlation in the squares and cross products of standardized residuals up to the 15^{th} lag at a confidence level of 5%. Panel B: NL, MAE, RMSE and R^2 are multivariate versions of the standard univariate negative log-likelihood statistic, the mean absolute error, the root mean squared error and the R^2 statistics between simulated values and predicted values of conditional covariances, respectively.

Panel A

Model	% variance of stand. res. in CI	% Ljung-Box rejected
CCC-GARCH	100 (2/2)	0 (0/4)
DCC-GARCH	100 (2/2)	0 (0/4)
RS-DCC-GARCH	100 (2/2)	0 (0/4)
CCC-TreeGARCH	100 (2/2)	0 (0/4)
TreeDCC-TreeGARCH	100 (2/2)	0 (0/4)

Panel B

Model	IS-				OS-			
	NL	MAE	RMSE	R^2	NL	MAE	RMSE	R^2
CCC-GARCH	2071.7	0.1496	0.2255	0.6433	2175.5	0.1517	0.2210	0.6928
DCC-GARCH	2069.9	0.1487	0.2247	0.6423	2169.3	0.1511	0.2189	0.6931
RS-DCC-GARCH	2054.1	0.1497	0.2255	0.6423	2155.5	0.1518	0.2211	0.6910
CCC-TreeGARCH	1952.9	0.0684	0.1441	0.8759	2055.2	0.0699	0.1296	0.9155
TreeDCC-TreeGARCH	1951.9	0.0684	0.1421	0.8811	2053.8	0.0701	0.1271	0.9208

Table 2: Goodness-of-fit results from different model fits for a two-dimensional data example simulated from the general model (2.4) with threshold volatility and conditional correlation functions given in (3.1)-(3.3) and (3.2). Panel A: Tests on multivariate standardized residuals. Percentages of in-sample multivariate standardized residuals having variance in a confidence interval of one and percentages of rejected classical Ljung-Box tests investigating whether there is excess serial correlation in the squares and cross products of standardized residuals up to the 15^{th} lag at a confidence level of 5%. Panel B: NL, MAE, RMSE and R^2 are multivariate versions of the standard univariate negative log-likelihood statistic, the mean absolute error, the root mean squared error and the R^2 statistics between simulated values and predicted values of conditional covariances, respectively.

Panel A

Model	% variance of stand. res. in CI	% Ljung-Box rejected
CCC-GARCH	100 (2/2)	0 (0/4)
DCC-GARCH	100 (2/2)	0 (0/4)
RS-DCC-GARCH	100 (2/2)	0 (0/4)
CCC-TreeGARCH	100 (2/2)	0 (0/4)
TreeDCC-TreeGARCH	100 (2/2)	0 (0/4)

Panel B

Model	IS-				OS-			
	NL	MAE	RMSE	R^2	NL	MAE	RMSE	R^2
CCC-GARCH	2409.8	0.1152	0.1812	0.6154	2592.4	0.1368	0.2230	0.5908
DCC-GARCH	2405.5	0.1042	0.1646	0.7661	2584.4	0.1237	0.1995	0.7683
RS-DCC-GARCH	2399.5	0.1155	0.1815	0.6088	2580.8	0.1368	0.2230	0.5896
CCC-TreeGARCH	2313.7	0.0682	0.1281	0.7228	2491.1	0.0844	0.1665	0.6837
TreeDCC-TreeGARCH	2307.2	0.0533	0.1053	0.8997	2482.4	0.0663	0.1352	0.8903

Table 3: Goodness-of-fit results from different model fits for a two-dimensional data example simulated from the general model (2.4) with threshold volatility functions (3.1)-(3.3) and conditional correlations following Engle's DCC(1,1) model with parameters given in in (3.4). Panel A: Tests on multivariate standardized residuals. Percentages of in-sample multivariate standardized residuals having variance in a confidence interval of one and percentages of rejected classical Ljung-Box tests investigating whether there is excess serial correlation in the squares and cross products of standardized residuals up to the 15^{th} lag at a confidence level of 5%. Panel B: NL, MAE, RMSE and R^2 are multivariate versions of the standard univariate negative log-likelihood statistic, the mean absolute error, the root mean squared error and the R^2 statistics between simulated values and predicted values of conditional covariances, respectively.

Panel A

Model	% variance of stand. res. in CI	% Ljung-Box rejected
CCC-GARCH	100 (50/50)	5.32 (133/2500)
DCC-GARCH	100 (50/50)	5.32 (133/2500)
RS-DCC-GARCH	100 (50/50)	5.32 (133/2500)
CCC-TreeGARCH	100 (50/50)	5.52 (138/2500)
TreeDCC-TreeGARCH	100 (50/50)	4.96 (124/2500)

Panel B

Model	IS-				OS-			
	NL	MAE	RMSE	R^2	NL	MAE	RMSE	R^2
CCC-GARCH	30532	0.0809	0.1167	0.8701	30332	0.0728	0.1215	0.8912
DCC-GARCH	30532	0.0809	0.1167	0.8701	30332	0.0728	0.1215	0.8912
RS-DCC-GARCH	29290	0.0965	0.1383	0.7445	31030	0.0840	0.1379	0.7629
CCC-TreeGARCH	30305	0.0766	0.1133	0.9246	30179	0.0689	0.1231	0.9399
TreeDCC-TreeGARCH	30297	0.0767	0.1134	0.9252	30172	0.0691	0.1183	0.9492

Table 4: Goodness-of-fit results from different model fits for a fifty-dimensional data example simulated from the general model (2.4) with threshold volatility functions (3.1)-(3.3) and conditional correlations following the TreeDCC model with parameters given in in (3.5). Panel A: Tests on multivariate standardized residuals. Percentages of in-sample multivariate standardized residuals having variance in a confidence interval of one and percentages of rejected classical Ljung-Box tests investigating whether there is excess serial correlation in the squares and cross products of standardized residuals up to the 15^{th} lag at a confidence level of 5%. Panel B: NL, MAE, RMSE and R^2 are multivariate versions of the standard univariate negative log-likelihood statistic, the mean absolute error, the root mean squared error and the R^2 statistics between simulated values and predicted values of conditional covariances, respectively.

Model	Cond. corr. structure $\mathcal{R}_k$	Cond. corr. parameters c_k
TreeDCC	$X_{t-1,\text{DAX30}} \leq -0.8428$	0.7195
	$X_{t-1,\text{DAX30}} > -0.8428$ and $X_{t-1,\text{NIKKEI}} \leq -0.8236$	0.7446
	$X_{t-1,\text{DAX30}} > -0.8428$, $X_{t-1,\text{NIKKEI}} > -0.8236$ and $X_{t-1,\text{CAC40}} \leq 0.5082$	0.7406
	$X_{t-1,\text{DAX30}} > -0.8428$, $X_{t-1,\text{NIKKEI}} > -0.8236$ and $X_{t-1,\text{CAC40}} > 0.5082$	0.6884

Table 5: Estimation results for a multivariate time series of daily returns for nine equity indices. Data are for the in-sample time period between January 1, 1998 and December 29, 2000, consisting of 781 observations. Estimated conditional correlation structures and parameters are for a TreeDCC-TreeGARCH model fit.

Panel A

Model	% variance of stand. res. in CI	% Ljung-Box rejected
CCC-GARCH	100 (9/9)	16.0 (13/81)
DCC-GARCH	100 (9/9)	14.8 (12/81)
RS-DCC-GARCH	100 (9/9)	14.8 (12/81)
CCC-TreeGARCH	100 (9/9)	16.0 (13/81)
TreeDCC-TreeGARCH	100 (9/9)	13.6 (11/81)
F-MGARCH	100 (9/9)	14.8 (12/81)

Panel B

Model	IS-P				OS-P			
	NL	MAE	RMSE	R^2	NL	MAE	RMSE	R^2
CCC-GARCH	10144	1.4296	2.8271	0.0556	6241.3	1.8510	3.9284	0.0601
DCC-GARCH	10121	1.4323	2.8273	0.0517	6194.1	1.8560	3.9201	0.0594
RS-DCC-GARCH	9919.4	1.4259	2.8196	0.0658	6175.5	1.8459	3.9241	0.0641
CCC-TreeGARCH	10058	1.3805	2.7429	0.0832	6214.6	1.8665	3.8958	0.0898
TreeDCC-TreeGARCH	10053	1.3783	2.7363	0.0846	6170.8	1.8441	3.8726	0.0917
F-MGARCH	10138	1.3935	2.8081	0.0564	6238.9	1.8451	3.9727	0.0573

Table 6: Goodness-of-fit results from different model fits for a multivariate time series of daily returns for nine equity indices. Data are for the time period between January 1, 1998 and November 4, 2002, for a total of 1262 observations. Panel A: Tests on multivariate standardized residuals. Percentages of in-sample multivariate standardized residuals having variance in a confidence interval of one and percentages of rejected classical Ljung-Box tests investigating whether there is excess serial correlation in the squares and cross products of standardized residuals up to the 15^{th} lag at a confidence level of 5%. Panel B: NL, PMAE, PRMSE and PR^2 are multivariate predictive approximations of the standard univariate negative log-likelihood statistic, the mean absolute error, the root mean squared error and the R^2 statistics.

	EPA test statistic	
	range	semi-quadratic
1^{st} step:	3.5674 (0.006)	174.469 (0.014)
2^{nd} step:	3.0464 (0.020)	23.189 (0.087)
3^{rd} step:	2.1185 (0.088)	9.5524 (0.184)
4^{th} step:	2.0473 (0.076)	–
5^{th} step:	2.0333 (0.037)	–

Model	Worst performing index				
	1^{st} step	2^{nd} step	3^{rd} step	4^{th} step	5^{th} step
CCC-GARCH	-0.6698	1.2326	–	–	–
DCC-GARCH	-1.5376	0.8676	0.9829	0.9997	–
RS-DCC-GARCH	-1.0704	0.9999	1.0701	–	–
CCC-TreeGARCH	-1.5366	-0.6215	-0.4658	-0.0840	2.0333
TreeDCC-TreeGARCH	-1.8559	-1.3388	-1.3937	-3.4129	-2.0333
F-MGARCH	3.1723	–	–	–	–

Table 7: Testing differences of multivariate OS-PMSE performance terms among different multivariate GARCH models. Upper panel: Values of equal predictive ability (EPA) *t-type* tests using the range statistic T_R and the semi-quadratic statistic T_{SQ}. Corresponding P-values computed using a block-bootstrap procedure are given between parentheses. Lower panel: Worst performing index results for the construction of the confidence model sets. If the null hypothesis of EPA is rejected, the model with the largest worst performing index value is eliminated.

	EPA test statistic	
	range	semi-quadratic
1^{st} step:	4.9036 (0)	105.93 (0)
2^{nd} step:	3.7570 (0.001)	53.886 (0.001)
3^{rd} step:	3.5512 (0.009)	33.795 (0.002)
4^{th} step:	2.4459 (0.032)	11.771 (0.046)
5^{th} step:	1.6633 (0.098)	2.7665 (0.128)

Model	Worst performing index				
	1^{st} step	2^{nd} step	3^{rd} step	4^{th} step	5^{th} step
CCC-GARCH	0.8858	2.2975	–	–	–
DCC-GARCH	-1.7426	-0.0687	1.0873	2.0230	–
RS-DCC-GARCH	4.0886	–	–	–	–
CCC-TreeGARCH	-1.2793	-0.6559	0.3784	1.1503	1.6633
TreeDCC-TreeGARCH	-3.0435	-3.2785	-3.7332	-4.5061	-1.6633
F-MGARCH	1.9916	2.1327	2.1511	–	–

Table 8: Testing differences of multivariate OS-PMSE performance terms among different multivariate GARCH models. Upper panel: Values of equal predictive ability (EPA) *sign-type* tests using the range statistic T_R and the semi-quadratic statistic T_{SQ}. Corresponding P-values computed using a block-bootstrap procedure are given between parentheses. Lower panel: Worst performing index results for the construction of the confidence model sets. If the null hypothesis of EPA is rejected, the model with the largest worst performing index value is eliminated.