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Abstract

The paper develops an overlapping generations model with probabilistic aging of households. We define age as a set of personal attributes such as earnings potential, health and tastes that are characteristic of a person's position in the life-cycle. In assuming a limited number of different states of age, we separate the concepts of age and time since birth. Agents may retain their age characteristics for several periods before they move with a given probability to another state of age. Different generations that share the same age characteristics are aggregated analytically to a low number of age groups. The probabilistic aging model thus allows for a very parsimonious yet rather accurate approximation of demographic change and of life-cycle differences in earnings, wealth and consumption. Existing classes of overlapping generations models follow as special cases.

Keywords

Overlapping generations, aging, demographic change, life-cycle.

JEL Classification

D58, D91, H55, J21

1 Introduction

Many topics of fiscal policy and intertemporal macroeconomics are concerned with the behavior of overlapping generations (OLG) of households. Depending on the specific issues to be analysed, economists have a range of alternative models at hand. At one extreme end, one finds the representative agent model of infinitely lived consumers due to Ramsey (1928), Cass (1965) and Koopmans (1965). Weil (1987) has shown that this model can be supported by an OLG model if current generations are perfectly linked with future generations with an operative, altruistic bequest motive. Under this and a number of further conditions, the Ramsey model gives rise to the famous Ricardo-Barro (1974) debt neutrality theorem. If this perfect intergenerational link is cut either by the entry of new, unconnected generations at the extensive margin of population growth (Weil, 1989) or by a non-altruistic or non-operative bequest motive, then the debt neutrality theorem is violated and fiscal policy tends to redistribute across generations.[1]

At the other extreme is the two period OLG model, as pioneered by Samuelson (1958) and Diamond (1965), which gives rise to the standard crowding out hypothesis. The model is analytically very tractable but cuts down life-cycle detail to the bare bones. The length of a period featuring constant interest rates and constant rates of consumption etc. covers about thirty years in real time which imposes severe limits in quantitative empirical applications. At the expense of analytical tractability, empirical applications thus rely on numerically solved models with a large number of generations and detailed life-cycle patterns of earnings, consumption and savings. These models were pioneered by Auerbach and Kotlikoff (1987) and Hubbard and Judd (1987) for the analysis of tax and fiscal policy including demographic change and social security reform. More recent and refined applications include, among others, Altig and Carlstrom (1999), Altig, Auerbach, Kotlikoff, Smetters and Walliser (2001), and Imrohoroglu, Imrohoroglu and Joines (1999). Building on Ben-Porath (1967), Mincer (1974) and Heckman (1976), recent literature

[1] Smetters (1999) shows, however, that at least the long-run capital intensity is robust to most assumptions of the standard Ricardian model although neutrality tends to fail in the short-run.

such as Perroni (1995) or Heckman, Lochner and Taber (1998) have used OLG models with finite life-time to analyze the effects of taxes and various training subsidies on skill formation to endogenously explain life-cycle earnings profiles while Blundell, Costa Dias and Meghir (2003) have considered the effects of labor market policies on labor force participation over the life-cycle.

Life-cycle models with many generations capture detailed differences in wealth, marginal propensity to consume and in labor supply and earnings of different agents. But precisely for this reason, these models tend to be analytically intractable and must be solved numerically. They operate in a state space of as many as 108 dimensions as Laitner (1990) has shown. Therefore, they are very expensive to implement and the results are difficult to replicate. By way of contrast, the perpetual youth model due to Yaari (1965), Blanchard (1985) and Buiter (1988) is an analytically very tractable OLG model with a realistic period length. For this reason, and because of it's simple empirical application, it has become a widely used tool of quantitative analysis as well. The main drawback, however, is the rigid demographic assumption of an age independent mortality rate and the absence of life-cycle detail of earnings, consumption and savings.[2] For this reason, the model is not well suited for the analysis of aging or old age insurance. The recent extension by Gertler (1999) reconciles the perpetual youth model with an important aspect of life-cycle behavior by allowing for a stochastic transition from work to retirement and from retirement to death. It thereby opens up useful applications with a model that includes no more than two stock variables to represent the household sector. Although more complex, it is analytically tractable and easily implemented empirically. According to Cooley (1999), however, it is not really an improvement in quantitative terms over existing life-cycle models with many generations.

This paper develops an alternative approach that retains the simplicity of analytically aggregated OLG models and yet succeeds to replicate the rich life-cycle details of the

[2]Heijdra and Romp (2005), however, managed to include some life-cycle features in a continuous time version of the model that still allows for analytical solutions.

high dimensional finite horizon models. The model is easily implemented with a few state variables only and thereby makes it easier to replicate empirical studies on life-cycle effects of public policy and other shocks. The key idea is to separate the concept of "age" from "time since birth". We understand *age* as a set of physical or mental attributes such as earnings potential, health, tastes and other characteristics that may not change every period. This is the key difference to existing models, where aging and the passing of time are perfectly synchronized and age is understood as time since birth. Hence, there are as many states of age as there are life-cycle periods. An age period is identical to a time period. The passing of time is perfectly synchronized with aging. An OLG model with 55 life-cycle periods distinguishes 55 states of age to replicate as close as possible real world life-cycles. People age every single year.

Our alternative approach builds on the insight that, for empirical purposes, it is enough to distinguish only a few different states of age in order to capture empirically realistic life-cycle differences. Some people retain their health, earnings potential and youthfulness for many periods and thereby invite compliments such as "that person looks the same as ten years ago". Obviously, such a statement means that a person has not visibly aged. Other people age much faster, possibly as a result of accidents or simply due to personal health characteristics. Consequently, age and time since birth become very different. Since aging occurs stochastically, we label the proposed framework the probabilistic aging (PA) model. It will usually be sufficient to distinguish only a few different states of age, giving rise to the concept of an age group which consists of the generations or cohorts that find themselves in the same state of age. Choosing the number of age groups is a matter of how closely the empirical life-cycle features should be replicated.

In the PA model, people move stochastically from one age state to another with an age group specific transition rate. It is thus a natural generalization of the model by Gertler (1999). He distinguishes only two age groups that refer to workers and retirees. He also assumes, somewhat restrictively, that mortality risk sets in only after retirement while workers face no such risk. We allow for more age groups and find that about eight age states already allow for a close approximation of empirical life-cycle properties. Another generalization and extension is that the PA model allows for mortality already in younger age groups and thereby exhibits a closer approximation of demographic change. Furthermore, it is not only important to distinguish several worker groups to capture life-cycle earnings detail but also to distinguish between a few groups of retirees to take account of the substantial heterogeneity among old and very old generations. Health deteriorates and mortality increases rapidly among the very old generations.

In limiting the number of age groups, the state space of the PA model is drastically reduced as compared to standard life-cycle models with many generations. The PA model allows for analytical aggregation of generations into age groups and is, thus, easily implemented empirically. For example, the youngest age group could be chosen to correspond to the characteristics of 20 to 30 year olds. With agents normally stay several periods in an age group, age and time since birth become entirely different concepts. Although distinguishing only a few age groups, the PA model features a period length of one year and thereby supports realistic dynamic properties.[3] Another insightful feature of the PA model is that it can replicate a wide range of intertemporal household models as special cases by appropriately choosing the parameters governing the mortality and aging process.

The paper is organized as follows. Section 2 explains how an agent's life-cycle is modeled by defining age states with different characteristics such as earnings potential, mortality rate and possible other aspects as well. It is shown how the population is analytically aggregated into age groups. Section 3 solves the intertemporal optimization problem in the presence of mortality and aging risks. It also shows how various existing types of intertemporal models emerge as special cases of the PA model when choosing the aging and mortality process appropriately. Section 4 presents two illustrative empirical applications and section 5 concludes.

[3]The period length could also be a quarter if, for example, a higher frequency such as in real business cycle analysis is desired.

2 The Probabilistic Aging Model

In line with popular observations, we view aging as discrete events that occur stochastically. Some people remain young for many periods, with their labor productivity, health and other attributes being unchanged before an aging event moves them to the next state. The less fortunate ones grow old much more rapidly, maybe due to sickness, accidents and other key events. They might age every single year and arrive within a few periods at the final state of life before extinction. To capture these statements, one needs two different clocks to measure the passing of real time and the speed of the aging process. We measure real time in regular annual periods, with interest, prices and quantities appropriately defined per year. The aging clock runs slower and stochastically.

2.1 Life-Cycle Histories

We define a discrete number of A states of increasing age, and accordingly collect all agents with the same state of age $a \in \{1, \ldots, A\}$ in a corresponding age group a. People start life in state $a = 1$ with a given set of attributes. Aging means that the individual's characteristics change when she grows older by switching to state $a+1$. Figure 1 illustrates the change in the earnings potential as a result of aging where wages may first increase and later decrease. The smooth line introduces a familiar life-cycle pattern of wages that will result as an average outcome of very heterogeneous aging patterns.

Households differ not only by their date of birth, but also by their diverse life-cycle histories. An agent's life-cycle history is her biography of aging events that have happened since birth. It is represented by a vector α that records the past dates of aging events. At date t, the set of possible histories of a household that belongs to age group a is

$$\mathcal{N}_t^a \equiv \{(\alpha_1, \ldots, \alpha_a) : \alpha_1 < \ldots < \alpha_a \leq t\}. \tag{1}$$

A particular life-cycle history is represented by a vector $\alpha \in \mathcal{N}_t^a$. The element α_i, $i \in \{1, \ldots, a\}$, denotes the date at which the household who was formerly in age group

$i-1$, became a member of group i. In denoting the unborn by a virtual age group zero, the element α_1 lists the date of birth when an agent switches from the group of unborn to the first age group. We say that a member of group one aged only once with no further aging since birth. Nevertheless, different persons of the first age group are heterogeneous since they were born at different moments in the past. The set of possible biographies is $\mathcal{N}_t^1 = \{(\alpha_1) : t \geq \alpha_1\}$. By the same logic, α_2 is the date when an agent moved from group 1 into 2. People in age group 2 have aged twice. The set of life-cycle histories in this case is $\mathcal{N}_t^2 = \{(\alpha_1, \alpha_2) : \alpha_1 < \alpha_2 \leq t\}$. With this notation, the vectors α describing the biography of people in group a have a elements since such persons have aged a times in total.

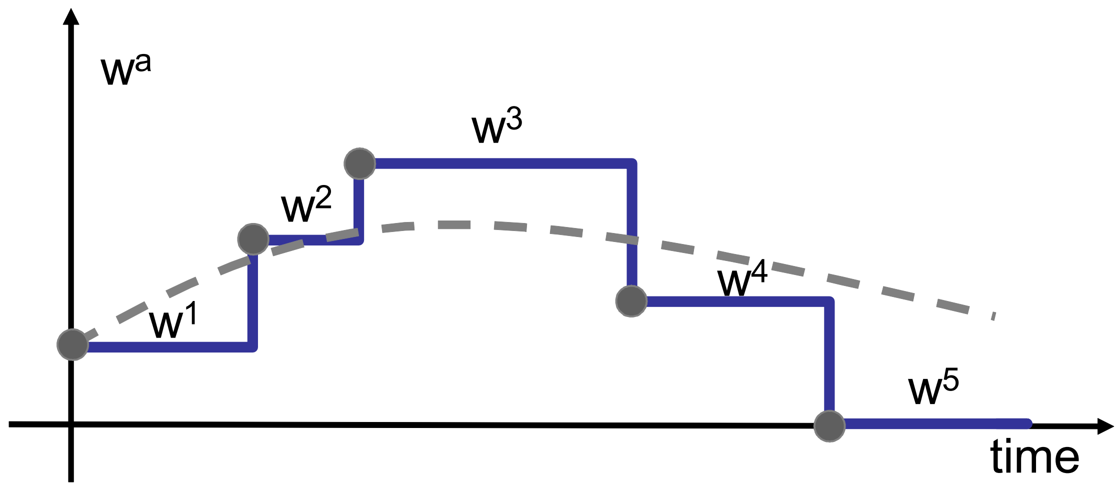

Fig. 1: Age Characteristics

The individual biographies are updated when a person is subject to an aging event. Suppose a person is in age group $a-1$ and is identified by a given biography $\alpha = (\alpha_1, \ldots, \alpha_{a-1})$. When the next aging event occurs in period t, she arrives in group a next period. Her biography is appended by the entry $t+1$ and will thus read $(\alpha_1, \ldots, \alpha_{a-1}, t+1)$. Accordingly, the set $\mathcal{N}_t^a$ of biographies of age group a will be augmented next period by all the biographies $\alpha' \in \mathcal{N}_t^{a-1} \times (t+1)$ that have $t+1$ as their last entry and refer to people who currently switch from group $a-1$ to a. Hence,

$$\mathcal{N}_{t+1}^a = \mathcal{N}_t^a \cup \mathcal{N}_t^{a-1} \times (t+1), \quad a \in \{1, \ldots, A\}. \tag{2}$$

Figure 2 illustrates the set of possible life-cycle histories of all agents that were born in period $\alpha_1 = t - 3$. Agents with a history along the vertical line remain in the most youthful group and were not subject to any further aging events since birth. Those with a history along the diagonal are aging most rapidly as they switch to a higher age group every period. At the top line, we see how all those born at the same date α_1 are stratified across age groups at date t as a result of probabilistic aging. In Figure 2, the short horizontal arrows indicate that our model also allows for mortality at any date in life. For simplicity, however, the numbers ignore mortality so that the horizontal sum of the top line adds up to 96 which is the size of the cohort at birth.

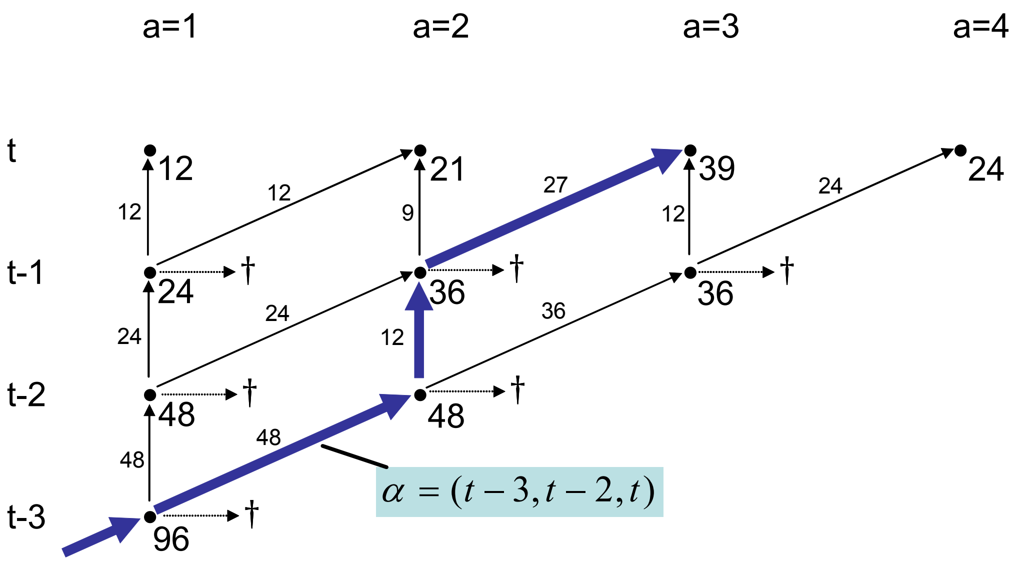

Fig. 2: Life-Cycle Histories

To model demographics, we allow for mortality among younger age groups. When an individual with an arbitrarily given life-cycle history plans for next period, she faces the risk of aging and dying. She must thus reckon with three possible events: (i) she dies with probability $1-\gamma^a$; (ii) she survives without aging and remains in the same age group with probability $\gamma^a\omega^a$, and (iii) she survives and ages and belongs to age group $a+1$ next period with probability $\gamma^a(1-\omega^a)$. Individuals in the last age group have exhausted the aging process and remain in this group with probability one, $\omega^A = 1$. They may either survive with probability γ^A within group A or die with probability $1-\gamma^A$. Observe that only the last age group behaves according to the mortality and demographic assumptions of Blanchard's (1985) perpetual youth model.

Since the characteristics of people such as their earnings potential differ across age groups, an agent's consumption, assets and other economic variables will generally depend on her particular life-cycle history. For example, assets depend on the agent's past earnings history which, in turn, is linked to her aging trajectory. To keep track of the population's heterogeneity, one must thus very carefully identify each agent by her age group as well as her aging biography α which also includes the date of birth. The number of agents at date t, in state of life a and with aging history α is given by $N^a_{\alpha,t}$. Within this group, agents are identical and face the same independent probability of moving to one of the alternative states. With stochastically independent risks, the law of large numbers implies that the above stated individual probabilities correspond to the fraction of people that are subject to the respective event. Consequently, the group is divided into three subgroups next period: (i) those who die and whose biography is updated to $\alpha^\dagger$, as indicated by the horizontal arrows in Figure 2; (ii) those who survive within the same age group a and move vertically; and (iii) those who are hit by an aging event, switch to the next higher age group and move diagonally in Figure 2. The age group and the biography α remains unchanged in case (ii) while switching to the next higher age group $a+1$ in case (iii) adds another event in a person's life-cycle history α and thereby results in a new biography α':

$$\begin{array}{llcll} (i) & N^\dagger_{\alpha^\dagger,t+1} & = & N^a_{\alpha,t}\cdot\left(1-\gamma^a\right), & \text{death,} \\ (ii) & N^a_{\alpha,t+1} & = & N^a_{\alpha,t}\cdot\gamma^a\omega^a, & \text{no aging,} \\ (iii) & N^{a+1}_{\alpha',t+1} & = & N^a_{\alpha,t}\cdot\gamma^a\left(1-\omega^a\right), & \text{aging.} \end{array} \tag{3}$$

2.2 Aggregate Demographics

People with the same biography are identical. At date t, age group a includes a number $N^a_{\alpha,t}$ of agents with the same biography α. The total number of people in age group a is obtained by adding up over all possible histories α ending up in this group a,

$$N_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} N_{\alpha,t}^a. \tag{4}$$

The aggregation formula (4) takes the sum over all possible biographies with varying dates of birth that could conceivably lead to age group a in period t. Figure 2 illustrates aggregation for a single cohort given by an identical date of birth α_1. From all 96 people born at date $\alpha_1 = t - 3$, only 39 end up until date t in age group $a = 3$. However, people have taken different aging trajectories to arrive in group 3, only one of them is drawn in bold. In general, an agent can reach group 3 only after aging twice. For example, age group 3 can be reached from date $t-2$ if the person ages in each single period, or starting at an earlier date if aging occurs less frequently. Group 3, however, cannot be approached from $t - 1$. In general, the most recent date of birth that can conceivably lead to age group a, is $\alpha_1 = t - a + 1$. Hence, for age group a, formula (4) implicitely takes the sum over all dates of birth $\alpha_1 \leq t - a + 1$ and subsequent biographies leading to group a.

It is assumed in (3) that all agents in a given group face identical transition probabilities. Given stochastically independent mortality and aging risks, one can now use the law of large numbers to aggregate the population into separate age groups. These aggregates evolve deterministically over time. One obtains

Proposition 1 *(a) The aggregate law of motion for age group $1 < a \leq A$ is*

$$N_{t+1}^a = \gamma^a \omega^a \cdot N_t^a + \gamma^{a-1} \left(1 - \omega^{a-1}\right) \cdot N_t^{a-1}, \quad \omega^A = 1. \tag{5}$$

(b) Age group 1 evolves according to

$$N_{t+1}^1 = \gamma^1 \omega^1 \cdot N_t^1 + N_{(t+1),t+1}^1. \tag{6}$$

(c) The total population grows by

$$N_{t+1} = N_t + N_{(t+1),t+t}^1 - \sum_{a=1}^{A} (1 - \gamma^a) N_t^a, \quad N_t \equiv \sum_{a=1}^{A} N_t^a. \tag{7}$$

Proof. (a) By definition, the stock of people in age group $a > 1$ in period $t+1$ is the current stock minus outflows plus inflows. The summation is over people identified by their aging biography α as in (4),

$$\begin{aligned}\sum_{\alpha\in\mathcal{N}_{t+1}^{a}} N_{\alpha,t+1}^{a} &= \sum_{\alpha\in\mathcal{N}_{t}^{a}} N_{\alpha,t}^{a} - \sum_{\alpha\in\mathcal{N}_{t}^{a}\times(t+1)} N_{\alpha,t+1}^{a+1} - \sum_{\alpha\in\mathcal{N}_{t}^{a}\times(t+1)} N_{\alpha,t+1}^{\dagger} + \sum_{\alpha\in\mathcal{N}_{t}^{a-1}\times(t+1)} N_{\alpha,t+1}^{a}, \\ N_{t+1}^{a} &= N_{t}^{a} - \gamma^{a}\left(1-\omega^{a}\right)N_{t}^{a} - \left(1-\gamma^{a}\right)N_{t}^{a} + \gamma^{a-1}\left(1-\omega^{a-1}\right)N_{t}^{a-1}.\end{aligned} \tag{8}$$

The first term on the right hand side is simply the current stock, as defined in (4). The second term is the outflow from group a to $a+1$. This flow is defined by the subset $\mathcal{N}_{t}^{a}\times(t+1)\subset\mathcal{N}_{t+1}^{a+1}$, which counts all agents with the last entry $t+1$ in their biography, meaning that they arrived in group $a+1$ exactly at date $t+1$ and not earlier. Since all agents in a face the same independent probability of the joint event surviving and aging, irrespective of the past biography (see 3.iii), the mass of the movers is given by the second term in the line below. The third term represents the outflow on account of death. The flow is given by the subset $\mathcal{N}_{t}^{a}\times(t+1)$ of the set $\mathcal{N}_{\alpha,t+1}^{\dagger}$ of all deceased. This subset consists of all biographies listing $t+1$ as a last entry and earlier entries $\alpha\in\mathcal{N}_{t}^{a}$. By the same arguments as before, a fraction $1-\gamma^{a}$ of the current age group is subject to the mortal event as in (3.i) which gives the term below in the second line. Finally, the last term is analogous to the second one, except that it applies to group $a-1$ and stands for an inflow to a. (5) follows by adding up the first three terms to $\gamma^{a}\omega^{a}N_{t}^{a}$ which corresponds to (3.ii). Since the last group cannot age further, the last event in (3) is precluded, implying the restriction $\omega^{A}=1$.

(b) The aggregation of group 1 is analogous. Replacing a by 1 in the last term of (8) gives $\sum_{\alpha\in\mathcal{N}_{t}^{0}\times(t+1)} N_{\alpha,t+1}^{1}$. The set of biographies $\mathcal{N}_{t}^{0}$ referring to unborns consists of an empty vector () since no aging event has occurred. The Cartesian product yields a set with a single element $\alpha=(t+1)$. Hence, the last term gives the number of newborns $N_{(t+1),t+1}^{1}$ flowing into group 1. Adding up the terms in the second line yields (6).

(c) Equation (7) follows by adding up (5) and (6) over all age groups. ■

The key demographic parameters are the birth rate, the transition rates to successive age groups, and the mortality rates. Since all are exogenous, the demographic subsystem is independent of economic influences and evolves autonomously according to (5-7). The demographic steady state results from the requirement that inflows and outflows of any age group must balance to yield constant group size. Using (5) and (6),

$$N^1 = \frac{N^1_{t,t}}{1-\gamma^1\omega^1}, \quad N^a = \frac{\gamma^{a-1}\left(1-\omega^{a-1}\right)}{1-\gamma^a\omega^a} \cdot N^{a-1}. \tag{9}$$

The stationary size of group 1 is determined by the exogenous inflow of newborns. The long-run magnitude of other groups and of the total population results upon recursively applying the second equation. For any demographic transition, the exogenous driving force is the flow $N^1_{t,t}$ of newborns. With this flow exogenously specified and constant, the system arrives at a stationary population N.

2.3 Wage Profiles

In standard OLG models with an annual aging process the population is divided into different vintages or cohorts identified by the year of birth. In these "cohort" models, individuals age every single period, moving along the diagonal line in Figure 2. There are thus as many age groups as cohorts, 55 as in the original Auerbach and Kotlikoff (1987) framework. The purpose of this section is to show that the PA model can incorporate life-cycle features in roughly the same detail as the annual cohort models even if one distinguishes only a few age groups. By drastically reducing the state space in this way, one can greatly simplify the quantitative, empirical analysis of life-cycle economies.

The empirical implementation of the PA model on real population data rests on the fact that it contains the annual cohort model as a special case. For simplicity, we consider a demographic stationary state and, thus, ignore time indices. Setting $\omega^a = 0$ in equation (5) implies that an aging event occurs with probability one in each period, leading to $\tilde{N}^t = \tilde{\gamma}^{t-1}\tilde{N}^{t-1}$. The concept of an age group thus becomes identical with a cohort or vintage where age t is measured by time since birth. The tilde indicates the decomposition in annual cohorts or population vintages $\tilde{N}^t$. We now take the age dependent survival rates

$\tilde{\gamma}^t$ from official mortality tables and construct the cohort composition of the population in a demographic steady state. Recursively applying $\tilde{N}^t = \tilde{\gamma}^{t-1}\tilde{N}^{t-1}$ yields the size of cohort t relative to the size of a new cohort. Summing up over all cohorts fixes the size of the new cohort compared to total population size N,

$$\tilde{N}^t = \tilde{N}^1 \prod_{s=1}^{t-1} \tilde{\gamma}^s, \quad \tilde{N}^1 = N \Big/ \sum_{t=1}^{T} \prod_{s=1}^{t-1} \tilde{\gamma}^s . \tag{10}$$

Taking a total length of life of T years, and based on actual survival rates, we have thus found the stationary decomposition of the population into a total of T cohorts or vintages. Alternatively, the total population may be decomposed into broader age groups N^a with each one containing several vintages,

$$\sum_{t=1}^{T} \tilde{N}^t = N = \sum_{a=1}^{A} N^a. \tag{11}$$

For the sake of concreteness, assume that total life-time consists of $T = 70$ periods as in Table 1 which considers the active population starting at age 20 and living until age 90. The total population is divided into a cross section of eight age groups. The first six are equally spaced and contain 10 cohorts each, the very old are subdivided into two smaller groups with five cohorts each. The first group contains vintages $N^1 = \sum_{t=20}^{29} \tilde{N}^t$. Line 4 of Table 1 gives the respective shares in the total population. Line 3 compares with the shares that would have resulted from actual, non-stationary population data.

The decomposition chosen in Table 1 rests on the idea that a representative agent spends an average period of 10 years in the first age group and then moves to the next one or dies. For the last two age groups, the average duration is assumed 5 years. Setting the date of birth at $\alpha_1 = 20$, this aging pattern uniquely determines the representative agent's life-cycle biography $\alpha = (20, 30, 40, \ldots, 80, 85)$ which serves as an aggregation key. In general, one may aggregate T cohorts into A age groups by

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_A), \quad \alpha_A < T, \quad N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{N}^t. \tag{12}$$

Our aggregation collects a subgroup of neighboring cohorts into an age group and takes the average wage and other attributes of these cohorts to define the characteristics of the age group. Lines 1 and 2 of Table 1, for example, collect cohorts 50-59 in age group 4. The biography of the representative agent who spends exactly the average duration in the age group and then moves to the next one, would always belong to the cohorts and age groups thus defined. In the PA model, however, very young who have aged fast, and very old people who have aged rather slow, may end up in the same age group. Taking age group 4 as an example, equation (12) imposes that the total mass of this age group is identical to the mass of cohorts 50-59 which defined the attributes of this group.

Table 1: Demographic and Life-Cycle Parameters

1. Age groups	1	2	3	4	5	6	7	8
2. Cohorts	20-29	30-39	40-49	50-59	60-69	70-79	80-84	85-89
3. Data N^a/N	0.168	0.222	0.192	0.168	0.120	0.089	0.025	0.016
4. Model N^a/N	0.179	0.177	0.175	0.168	0.148	0.107	0.031	0.016
5. Labor prod. θ^a	1.000	1.362	1.561	1.582	1.295	0.000	0.000	0.000
6. Prob. $1-\gamma^a$	0.001	0.001	0.004	0.012	0.028	0.042	0.096	0.200
7. Prob. $1-\omega^a$	0.099	0.099	0.096	0.089	0.074	0.061	0.115	0.000
8. Factor Ω^a	1.017	1.024	1.032	1.039	1.040	1.063	1.080	1.000
9. Propens. $1/\Delta^a$	0.045	0.049	0.056	0.067	0.083	0.108	0.166	0.227

Notes: θ^a life-cycle labor productivity determines wage $w^a = w\theta^a$, $1-\gamma^a$ probability of dying, $1-\omega^a$ probability of aging, Ω^a magnification interest factor reflecting increase in mortality, $1/\Delta^a$ marginal propensity to consume. Data sources: BFS (2004a), and own calculations.

Linking the aggregation of cohorts to a representative life-cycle history implies that an agent spends, on average, $\alpha_{a+1} - \alpha_a$ periods in age group a. In the above example, the average time spent in age group 4 would be 10 years. Given that the instantaneous probability of staying is $\omega^a\gamma^a$ according to (3.ii), the expected duration in group a is

$$\alpha_{a+1} - \alpha_a = 1/\left(1 - \omega^a\gamma^a\right). \tag{13}$$

The chosen age group decomposition of the actual population corresponds to a life-cycle history where an agent spends exactly the average duration in each age period. One can thus recover the demographic parameters of the PA model from our knowledge of aggregated population data N^a and duration in group a. In a demographic steady state, each age group must fulfill the restriction (5) which is

$$(1-\gamma^a\omega^a)\cdot N^a = \left(\gamma^{a-1}-\gamma^{a-1}\omega^{a-1}\right)\cdot N^{a-1}. \tag{14}$$

At this stage, one knows N^a from aggregated population data and $\gamma^a\omega^a$ from the age group duration as implied by the chosen aggregation. The only unknown in (13) is γ^{a-1} which is easily solved for all groups except the last one. For the last group, γ^A follows directly from (13) on account of the restriction $\omega^A = 1$. Lines 6 and 7 of Table 1 list the resulting values of the exit probabilities.[4] Figure 3 illustrates how these values approximate the true mortality rates from demographic data. The step function reflects the fact that mortality rates are identical for all agents within an age class.

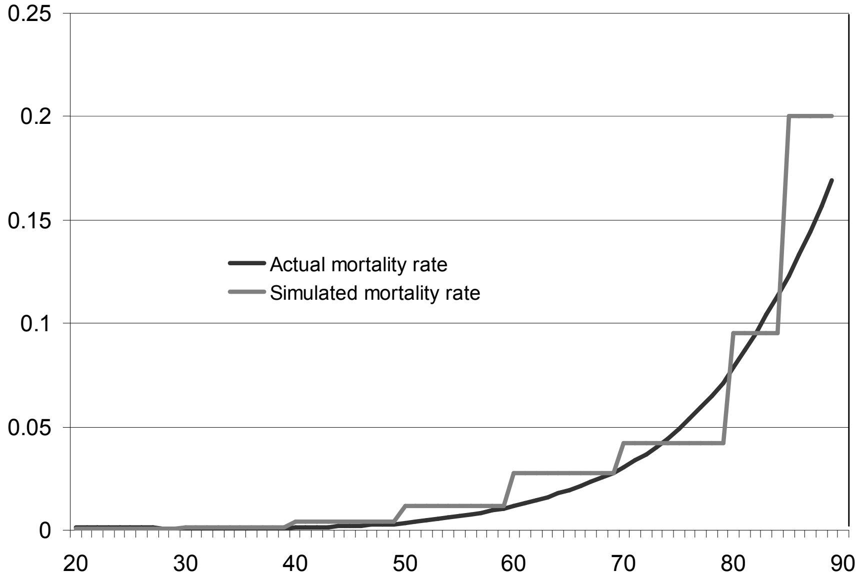

Fig. 3: Mortality Rates

As a next step, one uses actual wage data to find the life-cycle productivity parameters θ^a. For any given wage per efficiency unit of labor one obtains wages $w^a = w\theta^a$ as reflected in the step function of Figure 1. The life-cycle profiles of annual wages $\tilde{w}^t$ are readily

[4]The marginal propensity to consume in line 8 will be discussed in section 4.

available from cohort data. Condition (11) imposes that the total mass of an age group must be identical to the mass of cohorts which define the characteristics of this group. In perfect analogy, we now require that the aggregate wage bill of a given age group is the same as the accumulated wage bill from all cohorts that define this group:

$$w^a N^a = \sum_{t=\alpha_a}^{\alpha_{a+1}-1} \tilde{w}^t \tilde{N}^t. \tag{15}$$

This restriction determines the group specific wages w^a as an average of cohort specific wages $\tilde{w}^t$, taking the shares $\tilde{N}^t/N^a$ of the cohorts in that age group as weigths. Figure 4 illustrates how they approximate the empirical wage profile.

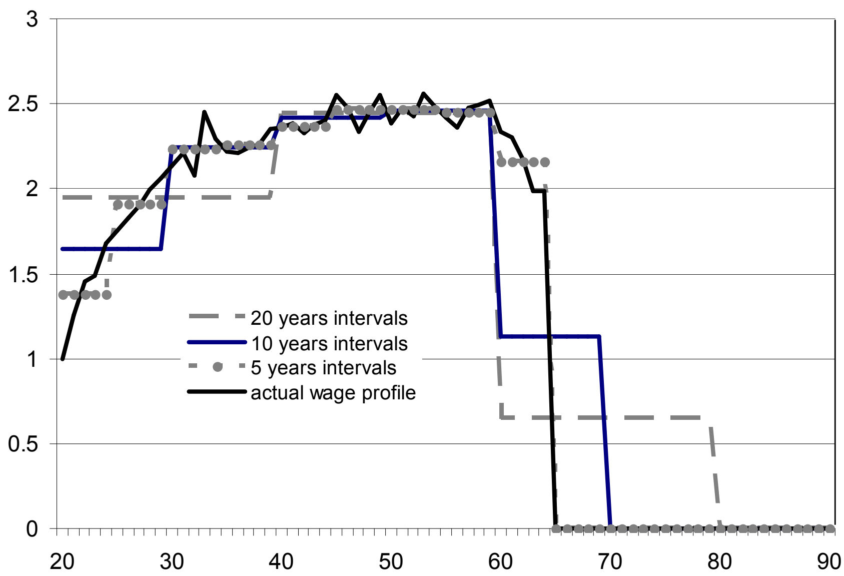

Fig. 4: Life-Cycle Wages. Data source: BFS (2004)

The solid, ragged line shows actual wage data for each cohort in Switzerland 2000. The solid step function with 10 year intervals corresponds to line 5 in Table 1. Note that the wage of age group 5 is an average of the relatively high wage prior to retirement and the zero wage after retirement when people are in their late sixties. The dashed line with 20 year intervals represents only 4 age groups with the first three comprising 20 cohorts and the last one 10 cohorts. Obviously, the approximation is rather crude for periods early and late in the life-cycle while it makes not much difference for agents in their mid-life.

Finally, the dashed line with 5 year intervals comes considerably closer to the true wage profile by taking averages over fewer cohorts. The gains in empirical approximation are highest for the youngest cohorts facing a steep wage profile, and for the cohorts close to retirement.

The step function stands for the representative agent who spends exactly ten years in each of the (earlier) age groups and then moves. This particular biography is, however, only one out of a huge number of alternative life-cycle histories that are possible with probabilistic aging. It will actually occur only with a tiny probability. What is then the wage that a new born agent may expect to earn in life-cycle period t, given that she is still alive in that period? After A periods, she may find herself in anyone of the age groups. The expected wage will thus depend on the probability of being in group a in period t which is, by definition, $P_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} P_{\alpha,t}^a$. An agent may take a multitude of alternative aging paths to arrive in state a at date t. However, looking back one period, she can approach state a from two possible states only. Either an agent was already in group a and remained there with probability $\gamma^a \omega^a$ or she was in group $a-1$ and switched from there with probability $\gamma^{a-1}(1-\omega^{a-1})$. The probability is thus stated recursively as

$$P_t^a = P_{t-1}^{a-1} \cdot \gamma^{a-1}\left(1-\omega^{a-1}\right) + P_{t-1}^a \cdot \gamma^a \omega^a, \quad P_t^a > 0 \Leftrightarrow 1 \leq a \leq \min\{t, A\}. \tag{16}$$

The probability P_t^a is zero if the age group number is larger than time since birth, $a > t$. The recursion starts with period 1 when the agent is born and belongs to the first group with probability $P_1^1 = 1$. Until period 2, she may have aged or not, giving $P_2^1 = P_1^1 \cdot \gamma^1 \omega^1$ and $P_2^2 = P_1^1 \cdot \gamma^1 (1-\omega^1)$. The recursion is easily continued.

The probability P_t of being alive in period t is the sum of probabilities taken over all age groups existing in that period. This probability is obviously one for a new born agent, $P_1 = P_1^1 = 1$. For all following periods, the probability is[5]

$$P_t \equiv \textstyle\sum_a P_t^a = \sum_a P_{t-1}^a \gamma^a. \tag{17}$$

[5]Substituting P_t^a from (16) into (17) gives $P_t = \left[P_{t-1}^1 \gamma^1 \omega^1\right] + \left[P_{t-1}^1 \gamma^1 \left(1-\omega^1\right) + P_{t-1}^2 \gamma^2 \omega^2\right] + \left[P_{t-1}^2 \gamma^2 \left(1-\omega^2\right) + P_{t-1}^3 \gamma^3 \omega^3\right] + \ldots + \left[P_{t-1}^{A-1} \gamma^{A-1} \left(1-\omega^{A-1}\right) + P_{t-1}^A \gamma^A \omega^A\right]$. Canceling terms and noting the restriction $\omega^A = 1$ yields (17).

The probability of being alive in state a of period t and then dying is $p_t^a = P_t^a \cdot (1-\gamma^a)$. These probabilities must sum up to unity.[6]

The probability of being in age group a, conditional on being alive in period t, is P_t^a/P_t. By the definition in (17), these probabilities sum up to one over all age groups. The period t wage, as expected at birth, conditional on being alive in period t, is

$$w_t = \sum_a \left(P_t^a/P_t\right) \cdot w^a. \tag{18}$$

Figure 5 plots the expected wage in period t against age group specific wages. The expected wage starts out at a value equal to the first wage class and then increases as people expect to move to the next higher wage group. In the first decile of Figure 5, the simulated expected wage is above the actual one since (18) takes an average over increasing wages. Within A periods, however, an agent starts to face a positive probability of already being in the last age groups with zero wages. As time passes, this probability ever increases so that the expected profile falls below actual wages in annual cohort data as approximated by the step function. By way of contrast, the expected wage remains above the actual wage in periods late in the life-cycle. People have a positive chance of remaining young for several decades and thereby keeping high wages.

The PA model becomes identical to the cohort model if aging occurs periodically. The simulated wage profile can thus get arbitrarily close to the actual profile if aggregation intervals are very short. However, the low dimensionality and simplicity of the PA model would be lost. How must one then interpret the deviation of the average wage in the PA model from actual cohort data? Take people in age group 4 who earn the highest possible wage. A person born 30 years ago will belong to this age group only with some probability, and will have a good chance to be in another class with a lower wage. The expected wage must necessarily be lower than the wage of group 4. In the PA model,

[6]This is most easily checked in the case of age independent survival rates $\gamma^a = \gamma$. One then obtains $P_t = \gamma P_{t-1}$. The probability of surviving to period t and then dying emerges as $p_t = (1-\gamma) P_t = \gamma^{t-1}(1-\gamma)$. It is now verified that $\sum_{t=1}^{\infty} p_t = (1-\gamma)\left[1+\gamma+\gamma^2+\ldots\right] = 1$.

only part of the people in their 50s are actually in group 4. This is balanced by the fact that a fraction of people in their 20s, 30s and even 60s or 70s are in age group 4 as well. Therefore, as (15) states, the total wage bill of all agents in age group 4 of the PA model *exactly* corresponds to the aggregate wage of people in their fifties in the cohort model. For the same reason, the aggregate wage bill of agents in the last three age groups is zero by definition even though an agent 90 years old still has a positive probability of not being retired and earning a positive wage. That person would be in a lower class. Also, such a case occurs only with a small probability. Again, by (11), the mass of people actually belonging to the last group is the same as the mass of people in their late 80s in the cohort model. The PA model aggregates people according to their economic characteristics and not according to their time since birth!

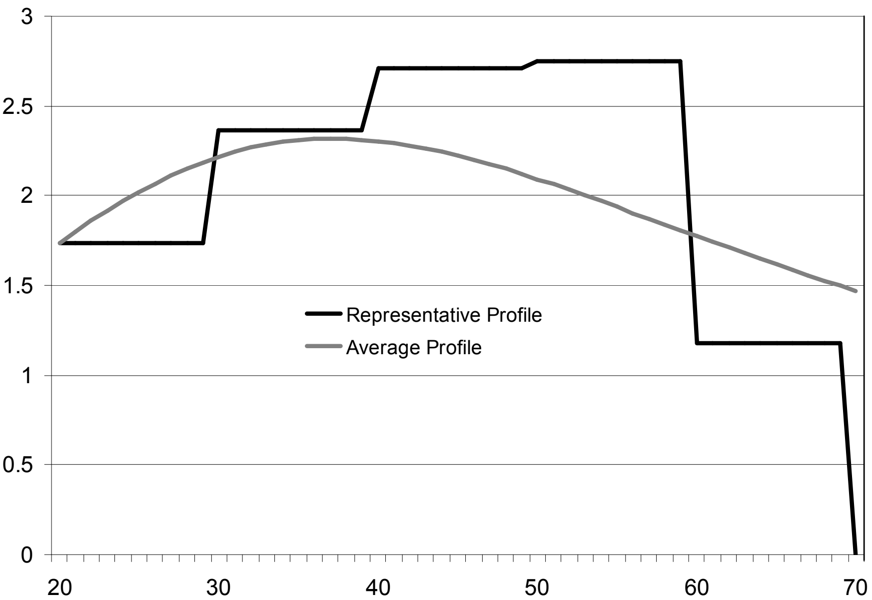

Fig. 5: Expected Life-Cycle Wages

Obviously, the aggregation of cohorts into broader age classes looses some detail. We can only look at the impact of economic shocks on age groups and not separately on different generations within this group. However, for most empirical purposes it is enough to distinguish people in their twenties, thirties and so on. There is probably no significant gain in distinguishing a 24 year old from a 25 year old for policy purposes. That distinction might also be illusory. One could even maintain that existing cohort models that distinguish agents by time since birth are unrealistic to some extent. Some people make careers faster than others. A 25 year old may earn exactly the same wage as a 35 year old. The two might be indistinguishable in economic terms. Cohort models would not take account of this extra heterogeneity. Further, depending on their past career paths, the wages of 35 year olds should be very heterogeneous. The cohort model would assign them all the same wage even though, within the same skill class, some 35 year olds have much higher wages than others of the same age since they were exceptionally lucky in advancing their career. Cohorts may be aggregated into broader age classes without significant loss in interpretation and empirical detail. The PA model is, however, much simpler to handle and much easier to implement empirically for quantitative purposes.

3 Life-Cycle Economies

The theory of probabilistic aging is now combined with an intertemporal general equilibrium model with life-cycle decisions and perfect foresight. In economic applications, a key distinction between age groups is the definition of wage related income,

$$y_t^a = \begin{cases} \left(1 - t^W\right) w_t \theta_t^a & : a \in \left\{1, \ldots, a^R - 1\right\}, \\ e_t & : a \in \left\{a^R, \ldots, A\right\}. \end{cases} \tag{19}$$

Ther are several states of work and retirement life. Wage related income per capita, y_t^a, differs across age groups but is identical for all agents within the same group. Importantly, it does not depend on life-cycle history α which is generally different for people in the same age group. Each worker in group a is thus endowed with θ^a efficiency units of labor which earn a wage of w_t per unit. Consequently, a worker's gross wage per unit of labor is $w_t^a = w_t \theta_t^a$ as illustrated in Figure 1. The government levies a proportional wage tax t^W to finance pensions and other public spending. Retirees receive a pension of e_t per capita from a pay-as-you-go (PAYG) social security system.

3.1 The Last Age Group

With perfect foresight, today's decisions must anticipate the consequences for future income and welfare. To take account of this interdependency, we start with the last age group and solve the model backwards. The last group is conceptually different from earlier ones because no further aging is possible ($\omega^A = 1$). Consequently, the set of possible events reduces to two. A person may either survive to the next period within the same age group, or die. The decisions of the last age group basically follow the principles of the perpetual youth model as in Blanchard (1985) albeit with one important difference. Since the last is only one of several age groups, average time spent in this group is rather limited. The mortality rate must thus be chosen much larger.

A member of the last age group is retired and receives an old age pension of $y_t^A = e_t$ per capita. Given current asset and pension income, agents choose consumption and savings and accumulate assets,

$$\gamma^A A_{\alpha,t+1}^A = R_{t+1} \left[A_{\alpha,t}^A + y_t^A - C_{\alpha,t}^A \right], \tag{20}$$

where C stands for consumption, A for assets, and $R = 1+r$ for the interest factor equal to one plus the annual interest rate. The value of assets is measured at the end of the period and earns, together with new savings $y - C$, an interest r_{t+1} until the end of next period. The term γ^A on the left-hand side arises due to the assumption of reverse life-insurance. In the absence of a bequest motive, the agent wants to get compensated during her life-time for any accidental bequests that she leaves upon death. Such compensation is assumed to be available from a competitive insurance sector. In fact, the same demand for insurance arises for any age group. Suppose the agent has assets of $S_{\alpha,t}^a$ at the end of period t, equal to the square bracket in (20). Aggregate end of period assets of this group are S^a. Since a fraction $1-\gamma^a$ dies, the insurance sector collects assets with a total value of $(1-\gamma^a) S^a$. On the other hand, premiums must be paid to those who survive, adding up to $\pi^a \gamma^a S^a$. The insurance sector breaks even with a premium of $\pi^a = (1-\gamma^a)/\gamma^a$ or $1+\pi^a = 1/\gamma^a$. If such insurance is available, the agent's assets next period are $A_{\alpha,t+1}^a = R_{t+1} (1+\pi_t^a) S_{\alpha,t}^a$ if she survives, or $\gamma^A A_{\alpha,t+1}^A = R_{t+1} S_{\alpha,t}^A$ as is stated in (20).

Preferences are represented by the CES non-expected utility theory proposed by Farmer (1990) and Weil (1990) which restrict individuals to be risk neutral with respect to variations in income but allow for an arbitrary intertemporal elasticity of substitution. Let β be a subjective discount factor reflecting the pure rate of time preference. Agents maximize expected welfare over the remaining life-time. The recursive formulation of intertemporally separable preferences yields the Bellmann equation

$$V\left(A_{\alpha,t}^{A}\right)=\max_{C_{\alpha,t}^{A}}\ \left[\left(C_{\alpha,t}^{A}\right)^{\rho}+\gamma^{A}\beta\cdot\left(V_{\alpha,t+1}^{A}\right)^{\rho}\right]^{1/\rho},\quad \sigma\equiv 1/\left(1-\rho\right). \tag{21}$$

Maximization is subject to (20). The CES parameter $\rho = 1 - 1/\sigma$ reflects the constant elasticity of intertemporal substitution σ. In defining

$$\eta_{\alpha,t}^{A}\equiv\frac{dV_{\alpha,t}^{A}}{dA_{\alpha,t}^{A}}\cdot\left(V_{\alpha,t}^{A}\right)^{\rho-1}, \tag{22}$$

one obtains optimality and envelope conditions

$$\left(C_{\alpha,t}^{A}\right)^{\rho-1}=\beta R_{t+1}\cdot\eta_{\alpha,t+1}^{A},\quad \eta_{\alpha,t}^{A}=\beta R_{t+1}\cdot\eta_{\alpha,t+1}^{A}. \tag{23}$$

The optimality condition reflects the usual tangency condition in intertemporal optimization that requires equality of the marginal rate of transformation and the agent's marginal rate of intertemporal substitution between current consumption and next period assets, $MRT = MRIS$. Suppressing indices other than t, $MRIS \equiv -dA_{t+1}/dC_t|_{dV=0} = C_t^{\rho-1}/\left[\gamma\beta\eta_{t+1}\right]$. The differential of the budget constraint yields $MRT \equiv -dA_{t+1}/dC_t = R_{t+1}/\gamma$. Equating them gives the optimality condition. Multiplying the envelope condition by βR_t, shifting the result by one period, and making use of the optimality condition yields the familiar Euler equation,

$$C_{\alpha,t+1}^{A}=\left(\beta R_{t+1}\right)^{\sigma}\cdot C_{\alpha,t}^{A}. \tag{24}$$

Desired consumption growth is a function of the expected interest rate relative to the subjective discount rate which is implicit in the factor β. A higher interest tilts the consumption profile towards the future, relative to a given life-cycle income profile, and thus implies higher savings. The sensitivity with respect to interest depends on the intertemporal elasticity of substitution. One can now obtain a closed form solution for consumption and indirect utility in terms of the agent's financial assets and the present value of her expected future pension benefits.

Proposition 2 *(a) The consumption function is*

$$C_{\alpha,t}^{A}=\left(1/\Delta_{t}^{A}\right)\cdot\left(A_{\alpha,t}^{A}+H_{\alpha,t}^{A}\right). \tag{25}$$

The inverse of the marginal propensity to consume, Δ_t^A, and human wealth equal to the present value of pension benefits, $H_{\alpha,t}^A$, are given by

$$\Delta_{t}^{A}=1+\beta^{\sigma}R_{t+1}^{\sigma-1}\cdot\gamma^{A}\Delta_{t+1}^{A},\quad H_{\alpha,t}^{A}=y_{t}^{A}+\gamma^{A}H_{\alpha,t+1}^{A}/R_{t+1}. \tag{26}$$

(b) Indirect utility over remaining life-time is

$$V_{\alpha,t}^{A}=\left(\Delta_{t}^{A}\right)^{1/\rho}C_{\alpha,t}^{A}=\left(\Delta_{t}^{A}\right)^{1/(\sigma-1)}\cdot\left(A_{\alpha,t}^{A}+H_{\alpha,t}^{A}\right). \tag{27}$$

Proof. (a) One must show that (25-26) satisfy the Euler equation (24). Replacing $C_{\alpha,t+1}^A$ in (24) by consumption in (25) yields $\gamma^A\left(A_{\alpha,t+1}^A+H_{\alpha,t+1}^A\right)=C_{\alpha,t}^A\cdot(\beta R_{t+1})^{\sigma}\cdot\gamma^A\Delta_{t+1}^A$. Replace the left-hand term by using (20) and (26), use again (25), and cancel out $C_{\alpha,t}^A$ from the result to get $\left(\Delta_t^A-1\right)R_{t+1}=\beta^{\sigma}R_{t+1}^{\sigma}\cdot\gamma^A\Delta_{t+1}^A$. Since this exactly corresponds to Δ_t^A in (26), we have shown that the Euler equation is identically fulfilled.

(b) Substitute (27) into the Euler condition and verify that the result satisfies the value function in (21). Substituting the left-hand side of the Euler equation yields $\beta\left(V_{\alpha,t+1}^A\right)^{\rho}=\beta^{\sigma}R_{t+1}^{\sigma-1}\Delta_{t+1}^A\cdot\left(C_{\alpha,t}^A\right)^{\rho}$. By (26), the first right-hand term is $\beta^{\sigma}R_{t+1}^{\sigma-1}\Delta_{t+1}^A=\left(\Delta_t^A-1\right)/\gamma^A$. Using this and $\left(V_{\alpha,t}^A\right)^{\rho}=\Delta_t^A\left(C_{\alpha,t}^A\right)^{\rho}$ as stated in (27) yields $\beta\gamma^A\left(V_{\alpha,t+1}^A\right)^{\rho}=\left(V_{\alpha,t}^A\right)^{\rho}-\left(C_{\alpha,t}^A\right)^{\rho}$, which corresponds to the value function in (21). ■

One may verify that the solution satisfies the intertemporal budget constraint that is obtained by solving forward the periodic budget in (20). The economics of intertemporal choice in the last age group is well known from the standard perpetual youth model.

3.2 Other Age Groups

The key difference to the last age group is that members of groups $a < A$, be it active workers or retired persons, are subject to the risk of aging while the last group is not. Hence, a person's expected utility next period, conditional on surviving, is

$$\bar{V}^{a}_{\alpha,t+1} \equiv \omega^{a} V^{a}_{\alpha,t+1} + (1-\omega^{a})\, V^{a+1}_{\alpha',t+1}. \tag{28}$$

With probability ω^{a}, the agent is not aging and expects welfare $V^{a}_{\alpha,t+1}$ next period. With probability $1-\omega^{a}$, she ages, switches to the next higher age group and expects welfare $V^{a+1}_{\alpha',t+1}$. In this event, the agent's biography must be updated as in (2) with an extra entry $t+1$. The Bellmann equation, referring to the optimization problem of an agent belonging to age group $a < A$, becomes

$$V\left(A^{a}_{\alpha,t}\right) = \max_{C^{a}_{\alpha,t}} \ \left[\left(C^{a}_{\alpha,t}\right)^{\rho} + \gamma^{a}\beta\left(\bar{V}^{a}_{\alpha,t+1}\right)^{\rho}\right]^{1/\rho}. \tag{29}$$

It is assumed, as in (20), that an actuarially fair, group specific insurance scheme is also available to younger age groups. Hence, maximization is subject to asset accumulation

$$\gamma^{a} A^{a}_{\alpha,t+1} = R_{t+1}\left[A^{a}_{\alpha,t} + y^{a}_{t} - C^{a}_{\alpha,t}\right], \quad A^{a}_{\alpha,t+1} = A^{a+1}_{\alpha',t+1}. \tag{30}$$

To solve for consumption policy, it is useful to define

$$\eta^{a}_{\alpha,t} \equiv \frac{dV^{a}_{\alpha,t}}{dA^{a}_{\alpha,t}}\left(V^{a}_{\alpha,t}\right)^{\rho-1}, \quad \bar{\eta}^{a}_{\alpha,t+1} \equiv \left[\omega^{a}\frac{dV^{a}_{\alpha,t+1}}{dA^{a}_{\alpha,t+1}} + (1-\omega^{a})\frac{dV^{a+1}_{\alpha',t+1}}{dA^{a+1}_{\alpha',t+1}}\right]\left(\bar{V}^{a}_{\alpha,t+1}\right)^{\rho-1}, \tag{31}$$

where the square bracket is a weighted shadow price of next period's assets that emerges due to the possibility of aging. With these definitions, one obtains the optimality and envelope conditions for consumption and assets,

$$\left(C^{a}_{\alpha,t}\right)^{\rho-1} = \beta R_{t+1}\cdot\bar{\eta}^{a}_{\alpha,t+1}, \quad \eta^{a}_{\alpha,t} = \beta R_{t+1}\cdot\bar{\eta}^{a}_{\alpha,t+1}. \tag{32}$$

The interpretation is much the same as in (23). The key difference is that the marginal rate of intertemporal substitution reflects the risk of aging. When the consumer postpones consumption and accumulates more assets, she compares the current utility loss with the *expected* utility gain next period as reflected in $\bar{\eta}$.[7] In deriving a modified Euler equation, it will be useful to state the marginal rate of substitution across age states which reflects the consumer's tradeoff between an extra Euro in the two alternative states. The differential of expected utility next period, $d\bar{V} = 0$, yields

$$\begin{aligned} MRS^a &\equiv dA^a/dA^{a+1}|_{d\bar{V}=0} = \tfrac{1-\omega^a}{\omega^a} \tfrac{dV^{a+1}/dA^{a+1}}{dV^a/dA^a} = \tfrac{1-\omega^a}{\omega^a} \left(\Lambda^a\right)^{1-\rho}, \\ \Omega^a &\equiv \omega^a \cdot [1 + MRS^a] = \omega^a + (1-\omega^a) \left(\Lambda^a\right)^{1-\rho}, \quad \Lambda^a \equiv \tfrac{V^{a+1}/C^{a+1}}{V^a/C^a}. \end{aligned} \tag{33}$$

The last equality uses (31) and $\eta^a = (C^a)^{\rho-1}$ from (32). The consumer would be willing to give up MRS^a Euros in state a if she could thus obtain an extra Euro in state $a+1$. The fact that an Euro saved yields an extra gain MRS^a in state $a+1$ acts like a magnification of the interest factor in the Euler equation. To see this, take out ω^a and $dV^a/dA^a = \eta^a (V^a)^{1-\rho} = (V^a/C^a)^{1-\rho}$ from the square bracket in (31) where the last expression used (31) and $\eta^a = (C^a)^{\rho-1}$ from (32). Therefore, $\bar{\eta}^a = \Omega^a \left[\bar{V}^a / (V^a/C^a)\right]^{\rho-1}$ with Ω^a given in (33). Writing expected utility as $\bar{V}^a = \left[\omega^a C^a + (1-\omega^a) \Lambda^a C^{a+1}\right] (V^a/C^a)$ results, upon substitution, in $\bar{\eta}^a = \Omega^a \left[\omega^a C^a + (1-\omega^a) \Lambda^a C^{a+1}\right]^{\rho-1}$. Using this in (32) yields the modified Euler equation where $\sigma = 1/(1-\rho)$,

$$\omega^a C^a_{\alpha,t+1} + (1-\omega^a) \Lambda^a_{\alpha,t+1} C^{a+1}_{\alpha',t+1} = \left(\beta R_{t+1} \Omega^a_{\alpha,t+1}\right)^\sigma \cdot C^a_{\alpha,t}. \tag{34}$$

Observe that probabilistic aging introduces a magnification $\Omega^a_{\alpha,t+1}$ of the interest factor. The following closed form solution for consumption and welfare is obtained.

[7] Suppressing α, we calculate the current utility loss from postponed consumption, $dV^a_t/dC^a_t = (V^a_t/C^a_t)^{1-\rho}$, and the expected utility gain per additional Euro tomorrow, $dV^a_t/dA^a_{t+1} = (V^a_t)^{1-\rho} \gamma^a \beta \bar{\eta}^a_{t+1}$. Since an Euro saved yields R_{t+1}/γ^a Euros tomorrow, the total gain from postponed consumption is $dV^a_t/dC^a_t = (R_{t+1}/\gamma^a) \cdot dV^a_t/dA^a_{t+1}$. Dividing through by the last term gives the equality of MRIS and MRT. Substituting the derivates yields the optimality condition in (32).

Proposition 3 *Optimal consumption policy $C^a_{\alpha,t}$ and indirect utility $V^a_{\alpha,t}$ are*

$$
\begin{aligned}
(i)\quad C^a_{\alpha,t} &= \left(1/\Delta^a_t\right)\left(A^a_{\alpha,t}+H^a_{\alpha,t}\right), \quad \sigma = 1/\left(1-\rho\right),\\
(ii)\quad V^a_{\alpha,t} &= \left(\Delta^a_t\right)^{1/\rho} C^a_{\alpha,t},\\
(iii)\quad \Delta^a_t &= 1+\gamma^a\beta^\sigma\left(\Omega^a_{t+1}R_{t+1}\right)^{\sigma-1}\Delta^a_{t+1},\\
(iv)\quad \Omega^a_{t+1} &= \omega^a+\left(1-\omega^a\right)\left(\Lambda^a_{t+1}\right)^{1-\rho}, \quad \Lambda^a_{t+1}=\left(\Delta^{a+1}_{t+1}/\Delta^a_{t+1}\right)^{1/\rho},\\
(v)\quad H^a_{\alpha,t} &= y^a_t+\gamma^a\bar{H}^a_{\alpha,t+1}/\left(\Omega^a_{t+1}R_{t+1}\right),\\
(vi)\quad \bar{H}^a_{\alpha,t+1} &= \omega^a H^a_{\alpha,t+1}+\left(1-\omega^a\right)\left(\Lambda^a_{t+1}\right)^{1-\rho}H^{a+1}_{\alpha',t+1}.
\end{aligned}
\tag{35}
$$

Δ^a_t is the inverse of the marginal propensity to consume and H denotes human capital equal to the present value of future wages and pension benefits.

Proof. We first show that the given consumption function indeed fulfills the Euler equation and is thus the optimal policy. Note that (35.ii) together with (33) implies Λ^a as in (35.iv). Since (35.ii-iv) depend only on the interest rate and other factors independent of α, the terms Δ^a_t and by implication Λ^a_t and Ω^a_t are all independent of α. Now insert (35.i) into the left hand term of (34), use $A^a_{\alpha,t+1}=A^{a+1}_{\alpha',t+1}$, collect terms, note the definitions of Ω^a_{t+1} and $\bar{H}^a_{\alpha,t+1}$, and get $\left[\Omega^a_{t+1}A^a_{\alpha,t+1}+\bar{H}^a_{\alpha,t+1}\right]/\Delta^a_{t+1}=\left(\beta\Omega^a_{t+1}R_{t+1}\right)^\sigma\cdot C^a_{\alpha,t}$. Multiply by $\gamma^a\Delta^a_{t+1}/\left(\Omega^a_{t+1}R_{t+1}\right)$ and use (35.v) and (30) on the left side: $A^a_{\alpha,t}+H^a_{\alpha,t}-C^a_{\alpha,t}=\gamma^a\beta^\sigma\left(\Omega^a_{t+1}R_{t+1}\right)^{\sigma-1}\Delta^a_{t+1}\cdot C^a_{\alpha,t}$. Substitute again (35.i) on the left hand side and cancel $C^a_{\alpha,t}$. The result corresponds exactly to (35.iii). Hence, the stated policy is optimal since it satisfies (34) which is merely a reformulation of the necessary conditions in (32).

The second step shows that (35.ii) gives indirect utility as it identically fulfills the Bellman equation. Again, start with the Euler equation and replace all consumption terms by the appropriate version of (35.ii). Multiply the result by $\left(\Delta^a_{t+1}\right)^{1/\rho}$, use the definitions of Λ^a_{t+1} and $\bar{V}^a_{\alpha,t+1}$ and get $\bar{V}^a_{\alpha,t+1}=\left(\beta\Omega^a_{t+1}R_{t+1}\right)^\sigma\cdot V^a_{\alpha,t}\left(\Delta^a_{t+1}/\Delta^a_t\right)^{1/\rho}$. Next, take the power of ρ, write $\rho\sigma=\sigma-1$, multiply by $\gamma^a\beta$, and use (35.iii) to substitute $\gamma^a\beta^\sigma\left(\Omega^a_{t+1}R_{t+1}\right)^{\sigma-1}\Delta^a_{t+1}=\Delta^a_t-1$. The result is $\gamma^a\beta\left(\bar{V}^a_{\alpha,t+1}\right)^\rho=\left(\Delta^a_t-1\right)\cdot\left(V^a_{\alpha,t}\right)^\rho/\Delta^a_t$. By (35.ii), $\left(V^a_{\alpha,t}\right)^\rho/\Delta^a_t=\left(C^a_{\alpha,t}\right)^\rho$, which is used on the right hand side. A minor rearrangement shows that the Bellman equation (29) is identically fulfilled. ■

Note that an agent cannot age any further, $\omega^A = 1$, if she is in the last age state. In that case, $MRS^A = 0$ and $\Omega^A = 1$ which yields the solution for the last age group as in proposition 2. Second, if all age classes face the same survival probability $\gamma^a = \gamma$ and thereby differ only in their earnings, then the marginal propensity to consume is invariant across age classes. This is most easily seen in case of a unit elasticity $\sigma = 1$ which yields $1/\Delta = 1 - \beta\gamma$ by (35.iii), but is also true more generally. If $\Delta^{a+1} = \Delta^a$, then Λ^a as well as Ω^a are both equal to one, implying $\Delta_t^a = 1 + \gamma\beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}^a$. Hence, Δ^a must indeed be the same for all groups. Age classes then differ only in terms of human wealth and consumption levels but all choose the same intertemporal consumption structure. This shows that the change in the marginal propensity to consume from one age class to the next exclusively reflects a change in the mortality rate.

Consider the most realistic case that the mortality rate increases between any two consecutive groups as in Table 1. Again, the Cobb Douglas case shows most transparently that the marginal propensity to consume $1/\Delta^a = 1 - \gamma^a\beta$ rises with a lower survival rate γ^a. Given that the ratio Δ^{a+1}/Δ^a falls below unity, and that $\rho < 0$ for an intertemporal substitution elasticity $\sigma < 1$, the factor Λ^a exceeds one and thereby raises Ω^a above one as well. Agents start to discount the future more heavily, at an effective rate $\Omega^a R$, as the end of life becomes a more probable event. Consequently, the marginal rate of substitution MRS^a as defined in (33) becomes larger when mortality is more common. With a higher probability of extinction, agents are less inclined to postpone consumption and thus consume a larger fraction of their resources immediately. Quite apparently, saving for the future makes less sense. This implies that the agent values consumption in state $a + 1$ relatively more than in state a which explains the increase in the marginal rate of substitution across age classes.

3.3 Aggregation

The simplicity and tractability of the simple perpetual youth with a constant mortality rate rests on the fact that it allows for analytical aggregation. The same applies to the PA model. The advantage of approximating actual demographic and life-cycle properties with a low dimensional system is possible only if age groups can be aggregated analytically. In section 2, we have argued that agents are completely identified by their biography α which also includes the date of birth as a first entry. In age group a, one records at date t a mass of agents $N_{\alpha,t}^a$ with the same life-cycle history. These agents are economically identical and all consume the same quantity $C_{\alpha,t}^a$. Aggregate consumption of age group a is obtained by summing over all possible biographies. Further adding over all age groups yields total, economy wide consumption,

$$C_t^a \equiv \sum_{\alpha \in \mathcal{N}_t^a} C_{\alpha,t}^a N_{\alpha,t}^a, \quad C_t \equiv \sum_{a=1}^{A} C_t^a. \tag{36}$$

The same principle applies to other static variables. When aggregating expressions that multiply with variables identical within an age group, or identical over the entire population, these constant terms drop out of the summation. For example, aggregating wage related income in (19) yields $\sum_\alpha y_t^a N_{\alpha,t}^a = y_t^a N_t^a$. Denoting labor supply in efficiency units by L^S and the retirees by N^R, one obtains

$$Y_t^a = y_t^a N_t^a, \quad Y_t = \left(1 - t^W\right) w_t L_t^S + e_t N_t^R, \quad L_t^S \equiv \sum_{a=1}^{a^R-1} \theta_t^a N_t^a, \quad N_t^R \equiv \sum_{a=a^R}^{A} N_t^a. \tag{37}$$

Similarly, when aggregating the consumption function of age group a stated in (25) and (35.i), we observe that the terms Δ_t^a, Λ_t^a and Ω_t^a are identical within each group. They implicitly depend only on the common interest rate and demographic parameters that differ across age groups but are constant within each group. Hence, aggregate consumption is simply $C_t^a = \left(A_t^a + H_t^a\right)/\Delta_t^a$.

Consider now the aggregation of dynamic variables such as human capital and asset wealth. Human capital is particularly simple in this basic model since wage related income and, thus, human capital per capita is the same for all persons within each group. This is most easily seen by solving forward (26) which yields the same present value of future income, discounted at a common rate, for all retirees. Writing the per capita value as $H_{\alpha,t}^A = h_t^A$, aggregate human capital is simply $H_t^A = h_t^A N_t^A$. The same holds for all earlier age groups as is obvious from (35). Hence,

$$H_t^a = h_t^a N_t^a, \quad h_t^a \equiv H_{\alpha,t}^a, \quad h_t^a = y_t^a + \gamma^a \cdot \frac{\omega^a h_{t+1}^a + (1-\omega^a)\left(\Lambda_{t+1}^a\right)^{1-\rho} h_{t+1}^{a+1}}{\Omega_{t+1}^a R_{t+1}}. \tag{38}$$

The law of motion for human capital per capita is repeated from (35.v-vi). The law of motion for the oldest age group follows on account of the restriction $\omega^A = 1$, implying $\Omega^A = 1$, and corresponds to the equation in (26). Note also that aggregate human capital obeys a different law of motion than per capita values since it is also driven by population dynamics leading to adjustments in group size N_t^a.

Deriving aggregate asset accumulation is more involved. Multiply (30) by $N_{\alpha,t}^a$ and sum over all agents according to their biography α. Use the aggregator (36) to get

$$\textstyle\sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t+1}^a \gamma^a N_{\alpha,t}^a = R_{t+1} \cdot S_t^a, \quad S_t^a \equiv A_t^a + Y_t^a - C_t^a, \tag{39}$$

where we have defined accumulated savings by S_t^a for convenient use below. Denoting the left hand term by X and multiplying it by ω^a leads to

$$\begin{aligned} \omega^a X &= \textstyle\sum_{\alpha \in \mathcal{N}_t^a} A_{\alpha,t+1}^a N_{\alpha,t+1}^a = A_{t+1}^a - \sum_{\alpha \in \mathcal{N}_t^{a-1} \times (t+1)} A_{\alpha,t+1}^a N_{\alpha,t+1}^a \\ &= A_{t+1}^a - \gamma^{a-1}\left(1-\omega^{a-1}\right) \textstyle\sum_{\alpha \in \mathcal{N}_t^{a-1}} A_{\alpha,t+1}^{a-1} N_{\alpha,t}^{a-1}. \end{aligned} \tag{40}$$

The first equality uses (3.ii). From all $N_{\alpha,t}^a$ people in age group a at date t, only a fraction $\gamma^a \omega^a$ survives and remains in the same group which corresponds to the sum of the first three terms in (8). The other part $1 - \gamma^a \omega^a$ represents outflows from the current age group due to aging or dying. The next equality expresses the fact that aggregate assets A_{t+1}^a of age group a next period is composed of the accumulated assets $\omega^a X$ of all those who are already in age group a in period t, plus the inflowing assets of all those who were in age group $a-1$ in period t, were hit by an aging event and are now part of group a. The subset flowing into age group a is the set of biographies which figure $t+1$ in the last entry, $\mathcal{N}_t^{a-1} \times (t+1) \subset \mathcal{N}_{t+1}^a$. The last equality looks more precisely at the newcomers in age group a. Corresponding to case (3.iii), the mass of newcomers is $N_{\alpha',t+1}^a = \gamma^{a-1}\left(1-\omega^{a-1}\right) N_{\alpha,t}^{a-1}$. Since all agents in group $a-1$ have the same chance of aging and surviving, the law of large numbers implies that an equal fraction $\gamma^{a-1}(1-\omega^{a-1})$ out of each class of biographies is moving to group a. Hence, the second line sums over the entire set $\mathcal{N}_t^{a-1}$, but takes only the common fraction of members in each biography group, and thereby precisely identifies the number of movers. Each of the movers has assets $A^a_{\alpha',t+1} = A^{a-1}_{\alpha,t+1}$. Performing now the aggregation in the second line, one finds $\sum_{\alpha\in\mathcal{N}_t^{a-1}} A^{a-1}_{\alpha,t+1} N^{a-1}_{\alpha,t} = (R_{t+1}/\gamma^{a-1}) S^{a-1}_t$ by applying (39) for age group $a-1$. Substituting the resulting expression for $\omega^a X$ into (39) yields aggregate asset accumulation of age group a,

$$A^a_{t+1} = R_{t+1}\left[\omega^a S^a_t + \left(1-\omega^{a-1}\right) S^{a-1}_t\right], \quad A^1_{t+1} = R_{t+1}\omega^1 S^1_t. \tag{41}$$

The survival factor γ^a cancels out in aggregate asset accumulation. By the insurance mechanism, the assets unintentionally left behind by the deceased are redistributed in full to the surviving fraction in each age group without any net loss in the aggregate. Note also that the newborns entering the first age group are bare of any assets since the unborn members of age group 0 cannot save by definition.

Finally, take the sum of (41) over all age groups and use the restriction $\omega^A = 1$. One finds that the savings of people switching from group $a-1$ to a add to aggregate assets in age group a but subtract from group $a-1$ so that these terms cancel in the aggregate. Consequently, total asset accumulation amounts to

$$A_{t+1} = R_{t+1}\left[A_t + Y_t - C_t\right], \quad A_t = \textstyle\sum_a A^a_t, \tag{42}$$

where Y is aggregate wage related income as given in (37).

3.4 A Synthesis of Models

The PA framework encompasses a wide range of intertemporal models as special cases and thereby shows how they are related by specific assumptions with respect to aging and demographic change. Since aging occurs infrequently and much slower than the passing of real time, one necessarily counts fewer age groups than cohorts. A cohort identifies agents by their age measured in terms of years since birth. An age group, in contrast, identifies people by their aging history, or by the number of aging events since birth. The two notions coincide only if real time and aging time are perfectly synchronized or if, in our discrete model, aging occurs annually.

With these definitions, we can now turn to the simplest case which is the Ramsey model with an infinitely lived representative agent. It emerges by excluding both aging and mortality, $\omega^a = \gamma^a = 1$ for all a which implies $\Omega^a = 1$ as well. With a constant population, savings and consumption result from the problem $V(A_t) = \max \left[C_t^\rho + \beta V(A_{t+1})^\rho\right]^{1/\rho}$ subject to $A_{t+1} = R_t \left[A_t + y_t - C_t\right]$. The standard decision rules are easily recovered from (35) under the given parameter restriction. Indices a and α become irrelevant. The marginal propensity to consume Δ_t and human wealth obviously remain independent of any age characteristics. Human wealth is the simple present value of future earings derived from $H_t = y_t + H_{t+1}/R_{t+1}$. The Euler equation (34) reduces to $C_{t+1} = (\beta R_{t+1})^\sigma C_t$ and requires the interest rate to satisfy $\beta R = 1$ in a stationary long-run equilibrium.

The perpetual youth model of Blanchard (1985) is a model with an infinity of cohorts but only one age group. It follows by setting $\gamma^a = \gamma < 1$ and $\omega^a = 1$. In any period, the event space is reduced to the first two events in (3) where the instantaneous mortality rate is invariant with respect to time and age. In the absence of aging, the Bellman equation in (29) becomes independent of age characteristics a. An agent's life-cycle history reduces to a single event which is birth at date $\alpha = \alpha_1$. Note that time since birth, $t - \alpha_1$, does not translate into aging which we define as a change in life-cycle characteristics. Hence, this model is quite appropriately termed a "perpetual youth" model. The constant mortality rate simply increases the effective discount factor to $\gamma\beta$. With $\omega^a = \Omega^a = 1$, the marginal propensity to consume, the inverse of (35.iii), is independent of age and the same for young and old generations since they all expect the same remaining life-time. However, it is higher compared to the Ramsey model and thereby reflects the finiteness of life. For the same reason, future wage income is less valuable since it might not be available in case of premature death, and is thus discounted at a higher effective rate. Human capital follows from $H_t = y_t + \gamma H_{t+1}/R_{t+1}$ and is independent of age, see (35.v-vi), since all generations receive the same wage. Agents differ only in the amount of financial assets that were accumulated since birth, and the level of consumption.

Extensions of the basic model have also made labor income dependent on time since birth $a = t - \alpha_1 + 1$ to approximate the empirical life-cycle pattern of wage income. While such extensions introduce some heterogeneity with respect to human wealth, they do not improve on the life-cycle pattern of the marginal propensity to consume which remains age independent as long as the survival rate remains constant. The PA model can reproduce this case by assuming annual aging, $\omega^a = 0$, where aging occurs each singe period. An age group is identical to a cohort so that the index for the age group measures time since birth. One need not keep track of life-cycle history separately. The Bellman equation simply becomes $V\left(A_t^a\right) = \left[\left(C_t^a\right)^\rho + \gamma\beta\left(V_{t+1}^{a+1}\right)^\rho\right]^{1/\rho}$. The decision rules are again recovered from (35). Since people necessarily age until next period, $\Delta_{t+1}^a = \Delta_{t+1}^{a+1}$ holds by definition, implying $\Lambda = 1$ and $\Omega = 1$. In fact, the marginal propensity to consume remains age independent as long as the survival factor is constant, $\Delta_t = 1 + \gamma\beta^\sigma\left(R_{t+1}\right)^{\sigma-1}\Delta_{t+1}$. Human wealth follows $H_t^a = y_t^a + \gamma H_{t+1}^{a+1}/R_{t+1}$ with y_t^a now depending on time since birth.

Gertler's (1999) extension of the perpetual youth model, recently applied by Keuschnigg and Keuschnigg (2004), is easily reproduced by allowing for only two aging events, birth and retirement, and two age groups. Agents are born as workers in age group one and then switch with constant probability into retirement. Mortality sets in only after retirement. The demographic assumptions are $\gamma^1 = 1$, $\gamma^2 < 1$, $\omega^1 < 1$ and $\omega^2 = 1$, leading to two age groups and an infinity of cohorts. This model not only introduces heterogeneity in human wealth. Retirees have a higher marginal propensity to consume than workers since they face a higher mortality rate. The current framework is a natural extension and generalization of this basic approach.

The PA model also encompasses the original Auerbach Kotlikoff (1987) model with an annual, although degenerate aging process. Each year since birth is divided into a separate age group a. In that sense, aging occurs every single period. The number of age groups thus coincides with the number of cohorts which is the number of life-cycle periods.

The life-cycle characteristics, measured in terms of earnings potential θ^a, change annually to reproduce realistic age earnings profiles until the fixed date of retirement approaches. Upon retirement, wage earnings drop to zero and are replaced by pension income. In the basic model, people are assumed to live with certainty until a fixed finite period and then die with certainty: $\gamma^a = 1$ for $a < A$ and $\gamma^A = 0$. The aging process is degenerate in the sense that people switch with certainty to the next higher age group every period, $\omega^a = 0$ all a, except $\omega^A = 1$. In particular, people switch to retirement at a fixed date with certainty when labor earnings drop to zero. This reproduces the *deterministic* segmentation between workers and retirees. Recent variations of the Auerbach Kotlikoff type OLG models also account for demographic change by deterministically shrinking the size of cohorts from one period to the next. The PA model would replicate this with $\gamma^a < 1$ for $a < A$ and $\gamma^A = 0$. These models still retain a fixed, deterministic time horizon and a fixed retirement date. Again, annual aging implies $\Lambda = 1$ and $\Omega = 1$. Time since birth fully describes an agent's biography which can thus be suppressed. Both human wealth and the marginal propensity to consume become age dependent. From (35), we recover $\Delta_t^a = 1 + \gamma^a \beta^\sigma (R_{t+1})^{\sigma-1} \Delta_{t+1}^a$ and $H_t^a = y_t^a + \gamma^a H_{t+1}^{a+1}/R_{t+1}$. Now, however, life-time ends with certainty after A years which introduces the end point restrictions $\Delta_t^A = 1$ and $H_t^A = y_t^A$ on account of $\gamma^A = 0$. In particular, the marginal propensity to consume increases up to unity when the last period of life approaches. The mortality rates and consequently the marginal propensities to consume increase rapidly as the end of life approaches. As Table 1 demonstrates, this feature can be approximated by the PA model when choosing shorter durations of average age periods for the last age groups.

Finally, the PA model collapses to the two period OLG model of Diamond (1965) and Samuelson (1958) when $\gamma^1 = 1$, $\gamma^2 = 0$, $\omega^1 = 0$ and $\omega^2 = 1$ and $y_{t+1}^2 = 0$. There is no life-time uncertainty ($\gamma^1 = 1$). People die with probability one after two periods, $\gamma^2 = 0$. Aging occurs deterministically after one period, $\omega^1 = 0$, when people switch to retirement with zero labor earnings, $y_{t+1}^2 = 0$. Agents are necessarily in age group 1 in their first period of life and in group 2 in their retirement period. Cohorts and age groups are thus exactly identical.[8] In this simplest case with periodic aging, people are fully identified by their age group which makes the indexing of life-cycle history superfluous. Agents are not endowed with any assets in their first period. This leaves the familiar problem of maximizing $V_t^1 = \left[(C_t^1)^\rho + \beta\left(C_{t+1}^2\right)^\rho\right]^{1/\rho}$ subject to $C_{t+1}^2 = R_{t+1}\left[y_t^1 - C_t^1\right]$.

3.5 General Equilibrium

Production of output Q_t is subject to a linear homogeneous, quasiconcave technology F using capital K_t and efficiency units of labor L_t as inputs. For realistic dynamics of investment in a small open economy, investment costs I^C are assumed to increase progressively with the rate of gross investment, reflecting installation costs of capital. The capital installation technology is again assumed linear homogeneous, leading to investment costs $I^C = \phi(I, K)$, where I is the amount of gross investment, and $\phi_I > 0$, $\phi_{II} > 0$, and $\phi_K < 0$. As a normalization, we may specify $\delta = \phi(\delta, 1)$, implying that $I^C = I$ in a steady state. Dividends χ_t are

$$\chi_t = Q_t - w_t L_t - I_t^C, \quad Q_t = F(K_t, L_t), \quad I^C = \phi(I, K). \tag{43}$$

Defining the end of period, cum dividend value of capital by V^K, optimal investment and employment policies follow from

$$V^K(K_t) = \max_{I_t, L_t} \ \chi_t + V^K(K_{t+1})/R_{t+1} \quad s.t. \quad K_{t+1} = I_t + (1-\delta)K_t. \tag{44}$$

Discounting with the households' interest factor R_{t+1} assures that the investors' no arbitrage-condition is satisfied, i.e. the return on dividend generating capital must equal the market rate of interest available on alternative assets. Denoting the shadow price of capital by $\lambda_t^K \equiv \partial V_t/\partial K_t$, the optimality and envelope conditions are

$$\phi_I = \lambda_{t+1}^K/R_{t+1}, \quad w_t = F_L, \quad \lambda_t^K = F_K - \phi_K + (1-\delta)\lambda_{t+1}^K/R_{t+1}. \tag{45}$$

[8] The case $\gamma^1 < 1$, $\gamma^2 = 0$, $\omega^1 < 1$ and $\omega^2 = 1$ would introduce a generalized two period OLG model with life-time uncertainty ($\gamma^1 < 1$) and second period labor income risk on account of probabilistic aging ($\omega^1 < 1$). Second period labor income would be y_{t+1}^1 with no aging and $y_{t+1}^2 < y_{t+1}^1$ with aging. Cohorts and age groups would not be identical anymore.

It is well known since Hayashi (1982) that the marginal and average shadow prices of capital are identical, $V_t^K \equiv \lambda_t^K K_t$. Finally, note that in a steady state, $I = \delta K$. Due to our normalization of the adjustment cost function ϕ, we have $\phi_I = 1$ and $\phi_K = 0$ in a steady state which yields the familiar condition $F_K = r + \delta$.

The model is closed by imposing the consolidated government budget constraint. Wage taxes must finance public consumption G which is kept constant per capita of the total population, and PAYG pension spending where L^S and N^R are given in (37),

$$t^W w_t L_t^S = e_t N_t^R + G_t, \quad G_t = g \cdot N_t. \tag{46}$$

Finally, we state the capital and labor market clearing conditions. In equilibrium, demand for efficiency units of labor corresponds to aggregate household sector labor supply, scaled by worker productivity. Further, accumulated household sector financial wealth absorbs the value of domestically issued equity, V_{t+1}, and foreign bonds D_{t+1},

$$L_t = L_t^S, \quad A_{t+1} = V_{t+1}^K + D_{t+1}. \tag{47}$$

When assets are perfectly substitutable, they must earn an identical rate of return equal to the market interest r_{t+1}. Given optimal household and firm behavior, and with all budget constraints fulfilled, Walras' Law implies the current account

$$D_{t+1} = R_{t+1} \left[D_t + Q_t - I_t^C - G_t - C_t \right], \tag{48}$$

where the stock of foreign net assets D_t is measured at the end of the period.

4 Illustrative Applications

4.1 A Calibrated Model

We calibrate the model to stylized data of a representative economy. Table 2 states key parameters that characterize preferences and technology. These are commonly used values as in Altig et. al (2001) and Blundell et al. (2003), for example, or in the real business cycle literature as in Baxter and King (1993) or King, Plosser, and Rebelo (1988). To keep the discussion short, we refer to these papers for a discussion and review of the econometric literature. The subjective discount rate is calibrated to replicate savings. The replacement rate gives the size of the pension compared to the last net wage income. A replacement rate of 70% is quite typical for the generous PAYG old age insurance systems in Europe. Tax revenue is used to finance government consumption and pension payments and requires a wage tax rate of 30%.

Table 2: Taste and Technology Parameters

Real interest rate r	0.050
Depreciation rate δ	0.100
Output elasticity of capital α	0.350
Subjective discount factor β	0.983
Intertemporal elasticity of substitution σ	0.400
Wage elasticity of labor supply ε^L	0.400
Retirement semielasticity ε^R	1.000
Proportional wage tax rate t^W	0.308
Pension replacement rate ζ	0.381

Notes: The labor supply elasticity is $\varepsilon^L \equiv \varphi'/(l\varphi'')$, the pension replacement rate $\zeta = e/[(1-t^W)w\theta^{a^R-1}]$, and the production function $Q = XK^\alpha(ZL)^{1-\alpha}$.

The parameters determining the life-cycle properties of the PA model have been discussed in Table 1 of section 2.3, except for the last two lines. Line 8 lists the factor Ω^a which reflects the agents' response to increasing finiteness of life over the life-cycle. The discussion of proposition 2 showed that the marginal propensity to consume rises as the end of life, in the sense of a shorter remaining expected life-time, approaches. Agents are less inclined to postpone consumption when mortality is high and when the fruits of savings might thus not be enjoyed. Consequently, agents value current consumption in state $a + 1$ relatively more than in state a where mortality is still low. The marginal rate of substitution between states a and $a+1$, measuring the amount of income to be given up in state a for an extra Euro of consumption and income in state $a+1$, is thus large when mortality in state $a+1$ is high compared to state a. This higher value of the marginal rate of substitution works like a magnification $\Omega^a > 1$ of the interest factor. Intuitively, agents start to discount the future more heavily, at an effective rate $\Omega^a R$, as the end of life becomes a more probable event. Inspecting the marginal propensity to consume in a steady state, $1/\Delta^a = 1 - \gamma^a \beta^\sigma (\Omega^a R)^{\sigma-1}$, one finds that the increasing finiteness of life in later age groups raises it for two reasons. First, the declining survival probability γ^a directly raises the consumption propensity. Second, this effect is magnified by the presence of $\Omega^a > 1$. If $1/\Delta^a$ rises for later age groups, then $\Delta^{a+1}/\Delta^a < 1$, implying $\Lambda^a > 1$ and $\Omega^a > 1$. Lines 8 and 9 of Table 1 clearly illustrate these life-cycle features. As is typical for OLG models, consumption propensities increase substantially as agents move to higher age groups, reflecting a shorter remaining expected life-time.

4.2 Economic Impact of Aging

4.2.1 Baby Boom

To demonstrate the usefulness and flexibility of the PA model, we choose two key scenarios of demographic change where the standard perpetual youth model is most restrictive. The first scenario mimicks the consequences of a temporary baby boom. We raise the inflow of newborns by 30 percent over a 20 year period, and then set it back to the original value in the initial demographic steady state. A demographic shock of this scale is not without precedent. According to Russell (1982), the number of births in the US amounted to 76 million over the period 1946-64 which is 17 million or 28.9% more than over the 18 years preceding the baby boom. With no other disturbances, the economy will eventually return to the initial equilibrium over a prolonged adjustment period.

Figure 6 illustrates how the PA model captures the demographic impact of the temporary baby boom. The first age group which collects all agents with characteristics of people in their twenties, rapidly grows larger. It attains its maximum size in period 2020 where it exceeds its initial size by 26%. Some of these newcomers age rather early and move to the next age group. With some delay, age group two with characteristics of people in their thirties starts to grow as well. This group attains maximum size somewhat later in period 2023 when it is 19% larger than in the initial equilibrium. As is evident from Figure 6, higher age groups grow even slower and take much longer to reach maximum size, and the increase compared to the initial steady state becomes smaller. The total population peaks in period 20 when the baby boom ends.

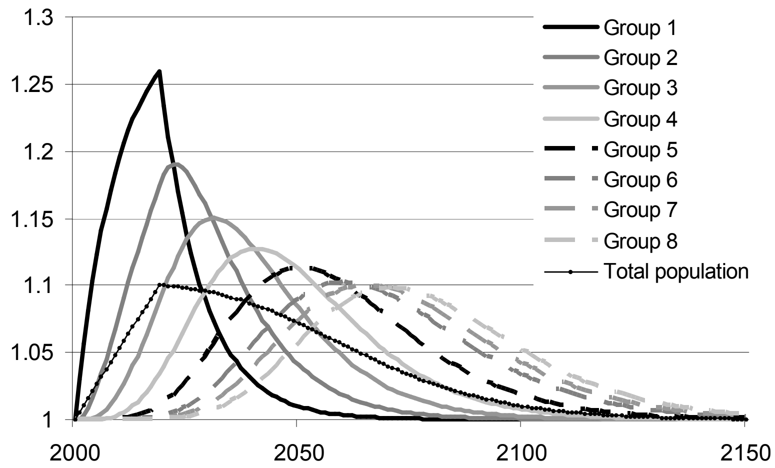

Fig.6: Demographic Impact of a Baby Boom

Figure 7 shows how a temporary baby boom might result in large swings in the net foreign asset position. In an open economy, net foreign assets reflect the imbalance between domestic savings and investment. Since domestic financial wealth equals the value of the domestic capital stock in the initial equilibrium, the net foreign asset position starts out from zero. Aggregate labor supply grows over a prolonged period and gets reversed again when the baby boom ends. Capital accumulation, and therefore the value of the capital stock as plotted in Figure 7, follow closely the same hump shaped pattern. However, due to the slow demographic process, it takes much longer until the increased labor income actually translates into an increase in aggregate financial wealth. Since the value of capital outpaces accumulated savings of domestic households, the difference must be borrowed on world capital markets, leading to a prolonged period of current account deficits and increasing net foreign debt. After the baby boom ends and the birth rate is back to normal, domestic capital stock declines again to follow the declining labor force. Aggregate assets continue to increase on account of the savings of the baby boomers which reverses the growth of net foreign debt. After slightly more than four decades, the net asset position turns positive and reaches a maximum after about two more decades. Thereafter, the economy monotonuously returns to the initial stationary equilibrium.

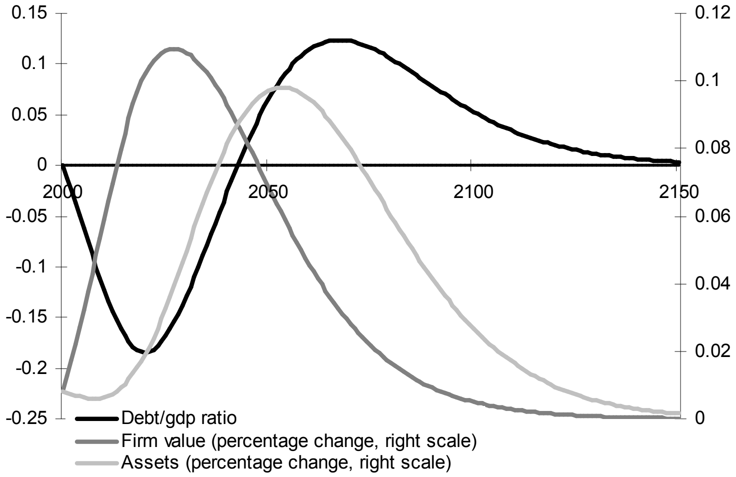

Fig. 7: Baby Boom and Net Foreign Assets.

4.2.2 Higher Life-Expectancy

The second scenario is an aging scenario where people become older on average and potentially live longer. The two statements are not identical. The first one means that the mass of people in their 80s becomes larger if more of the younger agents make it to their 80s. The second one means that the life-time horizon becomes longer. To implement the scenario, we raise the survival rates of age groups 5 to 8 by the factors given in line 4 of Table 3 but keep the expected duration in each age group constant. When people in their seventies become less mortal, they have a higher chance of reaching the next age group of people in their eighties.[9] Comparing columns 6 in Tables 1 and 3 shows that the

[9]People in group 6 are not literally in their seventies. Strictly speaking, group 6 consists of all young and old agents who share the economic characteristics of people in their seventies. However, the mass of people in group 6 is equal to the total mass of people belonging to cohorts 70-79 who define the

mortality rate $1-\gamma^a$ of this group declines from .042 to .023 which necessarily implies that a larger fraction of this group moves to the next one, instead of dying. The transition rate of aging $1-\omega^a$ increases from .061 to .079. Finally, since the last group becomes less mortal, $1-\gamma^8$ falling from .2 to .12, expected duration in that group rises from 5 to $(1-\gamma^8)^{-1} = 8.3$ years. The representative agent lives longer, 93.3 instead of 90 years.[10]

Table 3: Aging and Life-Expectancy

1. Age groups	1	2	3	4	5	6	7	8
2. Initial N^a	.179	.177	.175	.168	.148	.107	.031	.016
3. New N^a	.179	.177	.175	.168	.148	.121	.047	.058
4. Factor $\times\gamma^a$	1	1	1	1	1.01	1.02	1.05	1.1
5. Prob. $1-\gamma^a$	.001	.001	.004	.012	.018	.023	.050	.120
6. Prob. $1-\omega^a$	.099	.099	.096	.089	.083	.079	.158	0

Notes: $1-\gamma^a$ mortality rate, $1-\omega^a$ probability of aging.

Since this aging scenario keeps the inflow of newborns constant and a larger number of them makes it to their eighties or even nineties, the scenario also implies an increase in overall population size of 7.2 percent. According to official projections, the Swiss population is expected to largely remain constant in the long-run on account of a parallel decline in fertility. We suppress this part in order to observe the consequences of aging in isolation. Instead, we compare the aging scenario for different assumptions with regard to international capital mobility. For this purpose, we have calibrated the model with net foreign debt equal to zero. In a small open economy, the real interest rate is fixed on world markets. Any imbalance between domestic savings and investment is thus reflected in a change in the net foreign asset position. In a closed economy, the domestic real interest rate adjusts to keep net foreign assets to zero. This second scenario is interesting as aging

characteristics of group 6. See the discussion in section 2.3 and, in particular, equations (11) and (15).

[10] This last part corresponds to the aging scenario in Kalemli-Ozcan, Ryder and Weil (2000) who discuss the implications of longevity in a model with a single mortality rate.

is a worldwide phenomenon that might lead to a decline in the real interest rate.[11]

This aging scenario highlights the life-cycle properties of the model quite clearly. We first turn to the open economy case. Table 4 column 4 reports the long-run effects in the baseline scenario. With a constant real interest rate, the capital labor ratio and gross wages per efficiency unit of labor remain constant as well. Aging results in a larger number of old people which contributes to a larger population. In the absence of an offsetting decline in fertility, the mass of younger age groups remains almost unchanged, leading to a largely constant workforce and GDP. By assumption, the increased number of retirees erodes pensions per capita to an extent that keeps the pensions to GDP ratio constant. Note, however, that the increase in overall population by about 7% inflates government consumption and thereby necessitates a moderate increase in the wage tax by 1.3 percentage points.[12]

The higher tax erodes the net wage and thereby discourages hours worked at the intensive margin of labor supply.[13] This in itself would lead to lower employment and output. However, the large reduction in per capita pensions makes retirement less attractive compared to continued work. With δ rising from .45 to .57, postponed retirement raises aggregate labor supply at the extensive margin. For this reason, the aggregate workforce expands by more than 2%. This translates in a very small increase in *effective* labor supply for two reasons. First, active people work fewer hours and, second, the expansion of the work force occurs with the least productive people in age group 5. The net effect is a small increase in effective labor supply which is accompanied by an equally small increase in capital to keep the capital labor ratio constant. The last line clearly

[11]For example, computations by Börsch-Supan, Ludwig and Winter (2004, p.27) yield a decline in the return to capital by roughly one percentage point as a result of worldwide aging.

[12]Pension spending is a constant ratio of GDP, $eN^R = e_0 Y$, while public consumption is constant per capita, $G = gN$. Dividing (A.3) by the wage sum yields $t^W = e_0/(1-\alpha) + Ng/(wL)$. With wL largely constant, the wage tax rate must rise due to increasing N.

[13]The appendix introduces endogenous labor supply by allowing for variable hours worked and variable retirement. In age group 5, agents choose the "retirement date" δ, leading to income $y^5 = \delta\left(1-t^W\right) w\theta^5 l^5 + (1-\delta)\, e$, which reflects the extensive margin of labor supply.

demonstrates the life-cycle savings response. When people must expect a much reduced pension from the PAYG system (-18%), they must substantially increase their savings for retirement even though the interest rate remains constant. Aggregate assets are 31% higher. In the absence of domestic investment opportunities, this extra wealth is invested abroad and leads to a net foreign asset position of 71% of GDP in the long-run. The additional asset income from abroad allows for an increase in aggregate consumption by roughly four and a half percent, much higher than the increase in GDP. In column 5 we report the results from a simulation using a retirement semielasticity of 0.3, compared to 1 in the baseline case. Note that now fewer people decide to retire early, which leads to an overall decline in effective employment: The increase in the number of workers is now overcompensated by the decrease in labor supply due to lower net wages.

Table 4: Long-Run Impact of Aging

Key Macro Variables in %		Closed	Open	Open†
r	real interest* (0.050)	0.032	0.050	0.050
D^f/Y	debt GDP ratio* (0.000)	0.000	0.710	0.864
δ	retirement* (0.450)	0.037	0.569	0.493
t^W	contribution rate* (0.308)	0.323	0.321	0.325
e	pension p.c.	-13.416	-17.737	-22.060
$(1-t^W)w$	net wage	5.184	-1.918	-2.488
l	labor supply	2.042	-0.772	-1.003
N^W	workers	0.121	2.297	0.831
N^R	retirees	24.034	23.250	28.019
L^D	employment (effective)	2.159	1.390	-0.222
K	capital stock	17.837	1.390	-0.222
Y	gross dom.prod, GDP	7.394	1.390	-0.222
C	aggregate consumption	3.242	5.528	4.621
A	aggregate assets	16.385	30.763	34.954

Note: Pension/GDP ratio and public consumption per capita constant. %) Percent changes relative to ISS. *) Absolute values, initial values in brackets. †) Simulation with $\varepsilon = 0.3$.

Adjustment in the closed economy would be much different, as column "Closed" in Table 4 demonstrates. If the excess savings cannot be invested abroad, the real interest must decline, by 1.8 percentage points in Table 4, to balance savings and investment. With production much more capital intensive, wages rise substantially by 5.2% in the long-run and thereby stimulate labor supply both on the intensive and extensive margins. People work more hours (+2.0%) and they postpone retirement as work becomes more attractive relative to pension income. The expansion of effective employment in combination with higher capital intensity boosts the capital stock by 17.8% in the long-run, leading to an output gain of almost 8%. Keeping the pension to GDP ratio constant, the country thus affords an overall increase in pension expenditure which substantially cushions the decline in per capita pensions of 13.4%, instead of 17.7% in the open economy. The decline in the real interest rate in combination with a smaller decline in pensions per capita much reduces savings incentives and roughly halves the long-run increase in accumulated assets. Consequently, income from savings declines relative to the open economy scenario. Yet, aggregate consumption changes by almost the same amount. Higher wages substitute for lower capital income. Note that the wage tax rate now slightly declines. The expansion of the aggregate tax base is so large that it suffices to finance increased public spending.[14]

5 Conclusions

The theory of probabilistic aging separates the notion of age from time since birth. Economic age is defined by a set of personal characteristics such as labor productivity, tastes, health, mortality etc. that change less frequently over the life-cycle than periods since birth. People can retain their characteristics over several periods before they change as a result of aging. Aging occurs stochastically. Identifying a person with her biography which is the sequence of discrete aging events since birth, allows to analytically aggregate a multitude of different generations into a low number of age groups. The PA model

[14]The preceding footnote computes a tax rate of $t^W = e_0/(1-\alpha) + Ng/(wL)$. Total population and the wage bill wL now increase by roughly the same amount which allows for a constant tax rate.

is thus a generalized OLG model. Existing models follow as special cases which reflect specific assumptions with respect to aging frequency, mortality and period length.

In standard OLG models such as the most recent versions of the Auerbach and Kotlikoff (1987) framework, aging occurs every period and the probability of dying is unity in the last period of life. The number of life-cycle periods corresponds to the number of age groups or cohorts. As Laitner (1990) has shown, typical models with 55 generations operate in a state space of 108 dimensions and are thus very expensive to implement. If aging occurs less frequently, the number of age groups becomes smaller than the number of cohorts. We found that allowing for eight groups already leads to a quite accurate approximation of life-cycle properties. Counting as in Laitner (1990), the state space would be 14 instead of 108 dimensions. This low dimensionality greatly facilitates the empirical implementation and numerical solution of the model. It becomes much easier to replicate results from existing studies if necessary. First applications relating to a temporary baby boom and an increased longevity of life have demonstrated that the PA model is a much more powerful tool for quantitative empirical analysis of demographic change, compared to the perputual youth model due to Blanchard (1985) and Yaari (1968) as well as the recent extension by Gertler (1998) with two age groups. These models are popular because of analytical tractability and their simple empirical implementation. Yet they are quite limited when it comes to trace out the life-cycle implications of certain shocks or to investigate demographic change. The PA model is in between the Auerbach-Kotlikoff and Blanchard-Yaari models, allowing for analytical aggregation to a low number of age groups combined with a realistic modeling of the life-cycle and demographic change. Future work will be concerned with introducing labor market frictions and human capital formation to quantitatively explore the role of skill policies in offsetting the negative labor market consequences of aging and the fertility decline.

Appendix: Endogenous Labor Supply and Retirement

Consider age group $a = a_R$ with characteristics of people in their sixties. Assume that a fraction δ of this group is still active while the rest is retired. To retain symmetry, we assume that each agent receives a share δ of her labor income as wages and a share $1-\delta$ as an old age pension. A higher value of δ means postponed retirement which reflects the extensive margin of labor supply. We refer to δ as the retirement date since postponed retirement raises the share of wages in average labor income in that period. If actively employed, agents may work a variable number of l^a hours, reflecting the intensive margin. They trade off the income and consumption against disutility of work, both on the intensive and extensive margins. A particularly simple and yet realistic approach is to exclude income effects.[15] We thus assume an additively separable subutility over consumption and foregone leasure, $\bar{C}^a \equiv C^a - \delta\varphi(l^a) - \phi(\delta)$, where $\varphi(l)$ and $\phi(\delta)$ stand for convex disutility of work and postponed retirement, $\varphi', \phi' > 0$ and $\varphi'', \phi'' > 0$. Appropriately expanding (19) and (30) yields

$$\begin{array}{llcl}
(i) & V\left(A^a_{\alpha,t}\right) & = & \max_{C^a_{\alpha,t}, l^a_{\alpha,t}, \delta^a_{\alpha,t}} \left[\left(\bar{C}^a_{\alpha,t}\right)^\rho + \gamma^a\beta\left(\bar{V}^a_{\alpha,t+1}\right)^\rho\right]^{1/\rho} \quad s.t. \\
(ii) & \gamma^a A^a_{\alpha,t+1} & = & R_{t+1}\left[A^a_{\alpha,t} + \bar{y}^a_{\alpha,t} - \bar{C}^a_{\alpha,t}\right], \\
(iii) & \bar{C}^a_{\alpha,t} & = & C^a_{\alpha,t} - \delta_{\alpha,t}\varphi\left(l^a_{\alpha,t}\right) - \phi\left(\delta_{\alpha,t}\right), \\
(iv) & \bar{y}^a_{\alpha,t} & = & \max_{l^a_{\alpha,t}, \delta_{\alpha,t}} \delta_{\alpha,t}\left[w^a_t l^a_{\alpha,t} - \varphi\left(l^a_{\alpha,t}\right)\right] + \left(1-\delta_{\alpha,t}\right)e_t - \phi\left(\delta_{\alpha,t}\right), \\
(v) & y^a_{\alpha,t} & = & \delta_{\alpha,t} w^a_t l^a_{\alpha,t} + \left(1-\delta_{\alpha,t}\right)e_t, \quad w^a_t \equiv \left(1-t^W\right) w_t \theta^a_t.
\end{array} \tag{A.1}$$

A simple solution in three stages is possible. First, choose intensive and extensive labor supply to maximize effort adjusted wage income in (A.1.iv), yielding

$$w^a_t = \varphi'\left(l^a_t\right), \quad w^a_t l^a_t - \varphi\left(l^a_t\right) - e_t = \phi'\left(\delta_t\right). \tag{A.2}$$

Intensive labor supply exclusively depends on the current wage. The wage elasticity is determined by the curvature of φ. On the extensive margin, the chosen retirement date

[15] Excluding income effects is quite common in the literature on optimal income taxation or on real business cycles. See, for example, Heijdra (1998) and Greenwood et al. (1988) in intertemporal macroeconomics, Saez (2002) and Immervoll et al. (2004) on optimal income taxation, and Cremer and Pestieau (2003) for modeling postponed retirement this way.

reflects the difference between utility adjusted wages and alternative pension income. The more generous pensions are relative to net wages, the less people are inclined to incur the utility cost of postponed retirement, and the earlier they choose to retire. The share of wages in average labor income declines. Since wages and pensions are independent of history, labor supply and the retirement date are symmetric, $l_{\alpha,t}^a = l_t^a$ and $\delta_{\alpha,t} = \delta_t$. The same holds for average and effort adjusted labor income, y_t^a and $\bar{y}_t^a$, respectively.

The second step solves the intertemporal problem in (A.1.i-ii) of allocating subutility $\bar{C}_{\alpha,t}^a$ over time exactly as in Proposition 3, except that C is replaced by $\bar{C}$ and y by $\bar{y}$. The solution of the first two steps give optimal values for $\bar{C}$, l and δ. As a last step, it remains to compute optimal commodity consumption $C_{\alpha,t}^a$ by inverting (A.1.iii).

For all age groups $a < a^R$, the labor supply is reduced to the intensive dimension only. Consequently, $\delta = 1$ in (A.1) and ϕ is a constant that is set to zero. Finally, age groups $a > a^R$ are fully retired. Endogenous labor supply also leads to modification of government budget balance and labor market equilibrium,

$$t^W w_t L_t^S = e_t N_t^R + G_t, \qquad \begin{aligned} L_t^S &= \theta_t^{a^R} l_t^{a^R} \cdot \delta_t N_t^{a^R} + \textstyle\sum_{a=1}^{a^R-1} \theta_t^a l_t^a N_t^a, \\ N_t^R &= (1-\delta_t)\, N_t^{a^R} + \textstyle\sum_{a=a^R+1}^{A} N_t^a. \end{aligned} \tag{A.3}$$

References

[1] Altig, David and Charles T. Carlstrom (1999), Marginal Tax Rates and Income Inequality in a Life-Cycle Model, *American Economic Review 89*, 1197-1215.

[2] Altig, David, Alan J. Auerbach, Laurence J. Kotlikoff, Kent A. Smetters and Jan Walliser (2001), Simulating Fundamental Tax Reform in the United States, *American Economic Review 91*, 574-595.

[3] Auerbach, Alan J. and Lawrence J. Kotlikoff (1987), *Dynamic Fiscal Policy*, Cambridge University Press: Cambridge, MA.

[4] Barro, Robert J. (1974), Are Government Bonds Net Wealth?, *Journal of Political Economy 82*, 1095-1117.

[5] Baxter, Marianne and Robert King (1993), Fiscal Policy in General Equilibrium, *American Economic Review 83*, 315-334.

[6] Ben-Porath, Yoram (1967), The Production of Human Capital and the Life Cycle of Earnings, *Journal of Political Economy 75*, 352-365.

[7] BFS (2004), *Die Schweizerische Arbeitskräfteerhebung (SAKE),* Neuchâtel: Bundesamt für Statistik.

[8] Blanchard, Olivier J. (1985), Debt, Deficits and Finite Horizons, *Journal of Political Economy 93*, 223-247.

[9] Blundell, Richard, Monica Costa Dias and Costas Meghir (2003), *The Impact of Wage Subsidies: A General Equilibrium Approach*, Institute of Fiscal Studies and Bank of Portugal.

[10] Börsch-Supan, Axel, Alexander Ludwig and Joachim Winter (2004), *Aging, Pension Reform, and Capital Flows: A Multi-Country Simulation Model*, University of Mannheim (MEA).

[11] Buiter, Willem H. (1988), Death, Birth, Productivity Growth and Debt Neutrality, *Economic Journal 98*, 279-293.

[12] Cass, David (1965), Optimum Growth in an Aggregate Model of Capital Accumulation, *Review of Economic Studies 32*, 233-240.

[13] Cooley, Thomas F. (1999), Government Debt and Social Security in a Life-Cycle Economy. A Comment, *Carnegie-Rochester Conference Series on Public Policy 50*, 111-117.

[14] Cremer, Helmuth und Pierre Pestieau (2003), The Double Dividend of Postponing Retirement, *International Tax and Public Finance 10*, 419-434.

[15] Diamond, Peter (1965), National Debt in a Neoclassical Growth Model, *American Economic Review 55*, 1126-1150.

[16] Farmer, Roger E. A. (1990), Rince Preferences, *Quarterly Journal of Economics 105*, 43-60.

[17] Gertler, Mark (1999), Government Debt and Social Security in a Life-Cycle Economy, *Carnegie-Rochester Conference Series on Public Policy 50*, 61-110.

[18] Greenwood, Jeremy, Zvi Hercowitz and Gregory W. Huffman (1988), Investment, Capacity Utilization and the Real Business Cycle, *American Economic Review 78*, 402-417.

[19] Hayashi, Fumio (1982), Tobin's Marginal Q and Average Q: A Neoclassical Interpretation, *Econometrica 50*, 213-224.

[20] Heckman, James J. (1976), A Life-Cycle Model of Earnings, Learning, and Consumption, *Journal of Political Economy 84*, S11-S44.

[21] Heckman, James J., Lance Lochner and Christopher Taber (1998), Explaining Rising Wage Inequality: Explorations with a Dynamic General Equilibrium Model of Labor Earnings with Heterogeneous Agents, *Review of Economic Dynamics 1*, 1-58.

[22] Heijdra, Ben J. (1998), Fiscal Policy Multipliers: The Role of Monopolistic Competition, Scale Economies, and Intertemporal Substitution in Labour Supply, *International Economic Review 39*, 659-696.

[23] Heijdra, Ben J. and Ward E. Romp (2005), Old People and the Things that Pass: Shocks in a Small Open Economy, University of Groningen.

[24] Hubbard, Glenn and Kenneth L. Judd (1987), Social Security and Individual Welfare: Precautionary Saving, Borrowing Constraints, and the Payroll Tax, *American Economic Review 77*, 630-646.

[25] Immervoll, Herwig, Henrik Kleven, Claus Thustrup Kreiner und Emmanuel Saez (2004), *Welfare Reform in European Countries: A Microsimulation Analysis*, London: CEPR DP 4324.

[26] Imrohoroglu, Ayse, Selahattin Imrohoroglu and Douglas H. Joines (1999), Social Security in an Overlapping Generations Economy With Land, *Review of Economic Dynamics 2*, 638-665.

[27] Koopmans, Tjalling C. (1965), On the Concept of Optimal Growth, in: *The Econometric Approach to Development Planning*, Amsterdam: North-Holland.

[28] Kalemli-Ozcan, Sebnem, Harl E. Ryder and David N. Weil (2000), Mortality Decline, Human Capital Investment, and Economic Growth, *Journal of Development Economics 62*, 1-23.

[29] Keuschnigg, Christian and Mirela Keuschnigg (2004), Aging, Labor Markets, and Pension Reform in Austria , *FinanzArchiv 60*, 359-392.

[30] King, Robert, Charles Plosser, and Sergio Rebelo (1988), Production, Growth and Business Cycles: I. The Basic Neoclassical Model, *Journal of Monetary Economics 21*, 195-232.

[31] Laitner, John (1990), Tax Changes and Phase Diagrams for an Overlapping Generations Model, *Journal of Political Economy 98*, 193-220.

[32] Mankiw, Gregory N. and David N. Weil (1989), The Baby Boom, the Baby Bust, and the Housing Market, *Regional Science and Urban Economics 19*, 235-258.

[33] Mincer, Jacob A. (1974), Schooling, Experience and Earnings, New York: Columbia University Press.

[34] Perroni, Carlo (1995), Assessing the Dynamic Efficiency Gains of Tax Reform When Human Capital is Endogenous, *International Economic Review 36*, 907-925.

[35] Ramsey, Frank (1928), A Mathematical Theory of Saving, *Economic Journal 38*, 543-559.

[36] Russell, Louise B. (1982), *The Baby Boom Generation and the Economy*, Washington, D.C.: Brookings Institution.

[37] Saez, Emmanuel (2002a), Optimal Income Transfer Programs: Intensive Versus Extensive Labor Supply Responses, *Quarterly Journal of Economics 117*, 1039-1073.

[38] Samuelson, Paul A. (1958), An Exact Consumption Loan Model of Interest with or Without the Social Contrivance of Money, *Journal of Political Economy 66*, 467-482.

[39] Smetters, Kent (1999), Ricardian Equivalence: Long-Run Leviathan, *Journal of Public Economics 73*, 395-421.

[40] Weil, Philippe (1987), Love Thy Children: Reflections on the Barro Debt Neutrality Theorem, *Journal of Monetary Economics 19*, 377-391.

[41] Weil, Philippe (1989), Overlapping Families of Infinitely Lived Agents, *Journal of Public Economics 38*, 183-198.

[42] Weil, Philippe (1990), Nonexpected Utility in Macroeconomics, *Quarterly Journal of Economics 105*, 29-42.

[43] Yaari, Menahem E. (1965), Uncertain Lifetime, Life Insurance, and the Theory of the Consumer, *Review of Economic Studies 32*, 137-150.

[44] Weiss, Yoram (1986), The Determinants of Life Cycle Earnings: A Survey, in O. Ashenfelter and R. Layard (eds.), *Handbook of Labor Economics I*, Amsterdam: Elsevier, 603-639.