Why Disagreement May Not Matter (much) for Asset Prices

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Abstract

A simple consumption-based two-period model is used to study the (theoretical) effects of disagreement on asset prices. Analytical and numerical results show that individual uncertainty has a much larger effect on risk premia than disagreement if *(i)* the risk aversion is reasonably high and *(ii)* individual uncertainty is not much smaller than disagreement. Evidence from survey data on beliefs about output growth suggests that the latter is more than satisfied.

Keywords

riskfree rate, implied volatility, Survey of Professional Forecasters

JEL Classification

C42, G12, E44

1 Introduction

The effect of disagreement on asset prices is an important and contested topic. A growing literature explores the theoretical underpinnings for and empirical evidence of using disagreement as a new "risk factor."[1]

This (short and simple) paper uses a two-period endowment model (previously used by Varian (1985) and Giordani and Söderlind (2005)) to argue that basic asset pricing theory suggests that disagreement is not particularly important—at least not as long as the risk aversion is reasonably high and disagreement is of the same order of magnitude as individual uncertainty. The latter is backed up by evidence from the Survey of Professional Forecasters, which (uniquely) has information about both individual uncertainty and disagreement (about future GDP growth).

The basic reason for why disagreement matters for asset pricing is as follows. Disagreement implies that most investors get good deals on assets *they* (not others) believe will pay off. This leads to little diversification, that is, highly volatile consumption. Assets that are correlated with consumption therefore require substantial risk premia. However, with high risk aversion investors diversify to mitigate the volatility.

While the existing literature (see, for instance, the references in footnote 1) already show some of these results, it typically does so in much more complicated models (and often only implicitly). The contribution of this paper is to derive the results in a very simple model that produces easily interpretable results.

The outline of the paper is as follows: Section 2 presents the portfolio and consumption choice problem and discusses the main assumptions made to solve the model; Section 3 derives the asset pricing implications of the portfolio choice; Section 4 shows survey evidence on individual uncertainty and disagreement; and Section 5 concludes. Technical details are found in the appendices.

[1] For instance, Anderson, Ghysel, and Juergens (2005), Basak (2005), David (2008) and Shefrin (2005) analyse the effects in general equilibrium settings, Lintner (1969), Gerber and Hens (2006) and Fama and French (2007) use a CAPM setting, while Hong and Stein (2007) provides a general discussion of the link between price and trade volume dynamics.

2 Disagreement and Portfolio Choice

In this simple model, disagreement and individual uncertainty are kept distinct, markets are complete, there are only two periods and all investors are identical except for their beliefs about future GDP growth.

The objective of investor i is to maximise expected CRRA utility from consumption in periods 1 (C_{i1}) and 2 (C_{i2}). The latter is a random variable (function) of the state that will be realised in period 2. Let $f_i(s)$ be shorthand notation for investor i's probability density function (pdf) of the random state s. The objective is then to maximise

$$\frac{C_{i1}^{1-\gamma}}{1-\gamma} + \beta \int_s \frac{C_{i2}(s)^{1-\gamma}}{1-\gamma} f_i(s) ds. \tag{1}$$

(Clearly, the integral in the second term is the expected value of $C_{i2}^{1-\gamma}/(1-\gamma)$.) The budget constraints are

$$\begin{aligned} Y_{i1} &= C_{i1} + \int_s p(s) B_i(s) ds, \text{ and} \\ C_{i2}(s) &= Y_{i2}(s) + B_i(s) \text{ for all } s. \end{aligned} \tag{2}$$

The first equation is for period 1 and says that income Y_{i1} equals consumption, C_{i1}, plus the net investment in Arrow-Debreu (AD) assets: $p(s)$ is the market price for AD asset s and $B_i(s)$ is the number of such assets bought by investor i. The second equation, says that for every state s, consumption equals period 2 income, $Y_{i2}(s)$, plus the amount of AD asset s bought in period 1.

Investor i's first order condition for AD asset s is

$$p(s)^{1/\gamma} C_{i2}(s) = \beta^{1/\gamma} f_i(s)^{1/\gamma} C_{i1}. \tag{3}$$

Since this equation holds for every investor, it must also hold for a weighted average of them. Let g_i be the short hand notation for the cross-sectional pdf of different investors. Integrate the first order conditions across investors to get

$$p(s)^{1/\gamma} \int_i C_{i2}(s) g_i di = \beta^{1/\gamma} \int_i f_i(s)^{1/\gamma} C_{i1} g_i di. \tag{4}$$

Use the market clearing condition that output equals aggregate consumption in every state, $Y_2(s) = \int_i C_{i2}(s) g_i di$. For notational simplicity, let $Y_2(s) = s$ and solve for the price of asset s as

$$p\,(s) = \beta \left(\int_i f_i\,(s)^{1/\gamma} C_{i1} g_i\, d\,i \right)^{\gamma} / s^{\gamma}. \tag{5}$$

This AD price is driven by how abundant the state is, the preference parameters (β and γ), and by a weighted average the beliefs (pdfs) of the different investors (raised to $1/\gamma$) of the state. The weights in this average are $C_{i1} g_i$, which can be thought of as the "economic weight" of investor i, since it is the product of his/her consumption and relative frequency (cross-sectional pdf). We clearly need the consumption choice in period 1 (C_{i1}) of every investor, to find the asset prices.

In general, there is no explicit solution for the consumption choice—so we have to resort to numerical methods. However, in the special cases of logarithmic utility ($\gamma = 1$) or no disagreement, we know that all investors choose the same consumption in period 1 (if they have the same period 1 income), so the asset prices can readily be found. In other cases, we need to solve the model numerically or apply some sort of approximation. Given AD prices, it is straightforward to derive other asset prices.[2]

This paper tries to analyse the effect of disagreement on asset prices in a very simple setting—to get clear results. Therefore, I assume investors differ only with respect to their point forecasts of future output (income and preferences are the same across investors). In particular, investor i thinks that log income (GDP), $\ln Y_2$, is normally distributed, that is,

$$\text{investor } i \text{ believes that } \ln Y_2 \sim N(\mu_i, \sigma^2), \tag{6}$$

where μ_i is the point forecast (mean) of log output and σ^2 is the uncertainty. This means that $f_i(s)$ is a lognormal pdf.

I also assume that there is a continuum of investors who differ only with respect to their beliefs about the mean of $\ln Y_2$, μ_i. In particular, I assume that the cross-sectional distribution of these means is also normal, that is,

$$\text{the cross-section of investors is } \mu_i \sim N(\mu, \delta^2), \tag{7}$$

so δ^2 is a measure of disagreement (cross-sectional variance) about future aggregate out-

[2] For instance, a real bond pays 1 unit in every state, so its price is just the integral of $p(s)$—and the log real interest rate is the logarithm of 1/bond price. Alternatively, a claim on average consumption (a proxy for equity) pays s in state s, so its price is the integral of $p(s)s$—and the log return is the logarithm of s/price of the claim. Similarly, the price of a call option on average consumption with strike price K is the integral (from $s = K$ to infinity) of $p(s)(s - K)$.

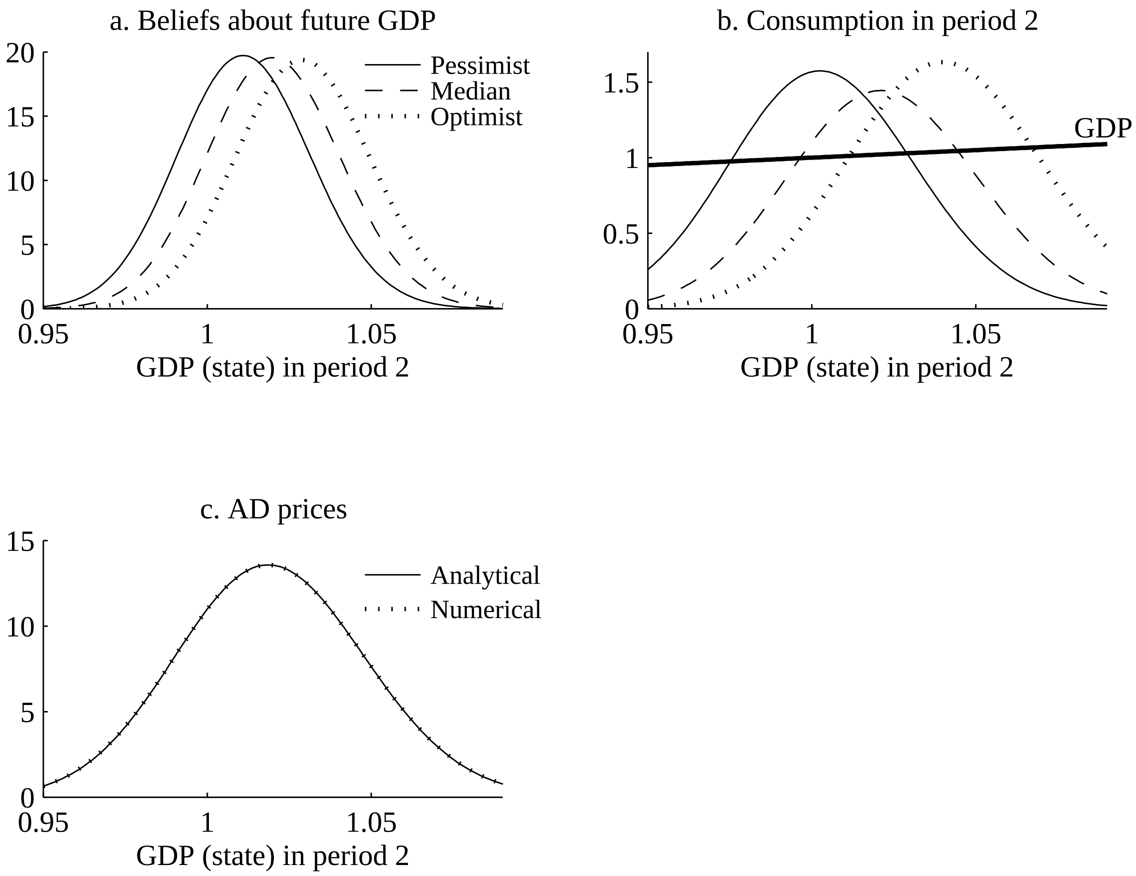

Figure 1: Beliefs, consumption plans and asset prices for logarithmic utility. This figure shows results based on $\mu = \sigma = \delta = 0.02$. The pessimist and optimist are the 0.33 and 0.67 percentiles in the cross-sectional distribution. See Appendix A for details on the numerical solution.

put. This means that g_i is a normal pdf. *Figure 1.a* illustrates the pdfs of three different investors.[3]

3 Asset Prices

3.1 Approximate Asset Prices

It is possible to derive approximate analytical results for the asset prices if we assume that all investors have the same period 1 consumption. I will later show that this provides

[3]Most models with heterogenous beliefs have only two agents. Here, a continuum of different beliefs turns out to simplify the analysis (besides being more realistic).

good approximations. In this case, the price of AD asset s is (see Appendix B)

$$p(s) = (\beta/s^{\gamma})(\omega/\sigma)^{1-\gamma}\varphi(s;\mu,\omega^2), \text{ with } \omega^2 = \sigma^2 + \delta^2/\gamma, \tag{8}$$

where first period income, Y_1, is normalised to unity and where $\varphi(s;\mu,\omega^2)$ denotes the lognormal pdf with $\mathrm{E}\ln s = \mu$ and $\mathrm{Var}(\ln s) = \omega^2$.[4]

The "variance" ω^2 combines individual uncertainty and disagreement, but the latter is divided by the risk aversion (γ). *Figure 1.c* illustrates the AD prices with a logarithmic utility function (for now, disregard the numerical results in the figure).[5]

The AD prices imply that the (log) real interest rate (r_f) and the cross-sectional (across investors) average expected excess return on a consumption claim ($\overline{\mathrm{E}}r_c - r_f$) are

$$r_f = -\ln\beta - (1-\gamma)\ln(\omega/\sigma) + \gamma\mu - \gamma^2\omega^2/2, \tag{9}$$

$$\overline{\mathrm{E}}r_c - r_f = (\gamma - 1/2)\omega^2. \tag{10}$$

(See Appendix B for derivations.) These are the standard expressions obtained from assuming normally distributed consumption growth (see, for instance, Campbell, Lo, and MacKinlay (1997), Ch. 8), except that ω^2 is the relevant measure of "uncertainty." As usual, the consumption claim is used as a proxy for equity. The riskfree rate is increasing in disagreement (at least if $\gamma > 1$) and the same is true for the equity risk premium (provided $\gamma > 1/2$). In addition, the equity risk premium is always increasing in the risk aversion coefficient (which is different from the result in David (2008)).

The AD prices also imply that option prices on the consumption claim follow the Black-Scholes formula, with "variance" ω^2. The basic reason is that future returns (consumption) are assumed to be lognormally distributed. In short,

$$\text{option prices follow Black-Scholes with "variance" } \omega^2. \tag{11}$$

(See Appendix B for derivations.)

The basic message of this paper is that ω^2 is (approximately) the relevant "uncertainty" for asset prices—and that disagreement is likely to play a relatively minor role

[4]Clearly, the pdf is $\varphi\left(s;\mu,\omega^2\right) = \exp[-(\ln s - \mu)^2/(2\omega^2)]/(s\sqrt{2\pi}\omega)$.

[5]Some general properties of the prices are as follows. First, states with high "probabilities" (pdf values)—according to most investors' beliefs—are more expensive. Second, comparing two states with the same probability, the poorer state is more expensive. Third, a higher risk aversion (γ) typically implies higher prices for poor states and lower prices for abundant states.

when risk aversion is reasonably high. For instance, if individual uncertainty is as large as the disagreement ($\sigma = \delta$) and the risk aversion (γ) is three, then only a quarter of the equity premium in (10) is due to disagreement. With higher risk aversion, it is even less. For instance, the equity risk premium in (10) is $\gamma\sigma^2 + \delta^2$ (plus a Jensen's inequality term), so individual uncertainty is scaled by the risk aversion while disagreement is not.

To explore this argument—and why disagreement matters at all for asset pricing—the next sections show exact solutions of the model for the two cases where that is possible (logarithmic utility or no disagreement) and numerical solutions for other cases.

3.2 Exact Asset Prices (Logarithmic Utility or No Disagreement)

The asset prices and returns in (8)–(11) actually hold exactly when the utility function is logarithmic ($\gamma = 1$). The reason is that all investors then choose the same consumption level in period 1 (see Rubinstein (1974) and Detemple and Murthy (1994)).[6] In this case, the relevant uncertainty (of a "representative investor") is $\omega^2 = \sigma^2 + \delta^2$, so individual uncertainty (σ^2) and disagreement (δ^2) are equally important.

With logarithmic utility, it is also straightforward to derive the consumption plan for investor i in each state, which helps understanding what disagreement implies for allocations and pricing. (See Appendix B for details.) *Figure 1.b* shows the consumption plans, $C_{i2}(s)$, of three different investors. Period 2 income, which would be the consumption for everyone if there were no disagreement, is also plotted. A consumption plan over (below) period 2 income in a state means that the investor buys (sells) that AD asset.

A pessimist ($\mu_i < \mu$) buys assets that pay off in states that he thinks are likely to happen and sells other assets—which seems intuitive. However, he actually overweights assets that pay off in poorer states than his point estimate. The reason is that those assets are in abundant supply (by more optimistic investors)—and the pessimist therefore thinks that their prices are particularly advantageous. (This sort of speculative behaviour is emphasised by David (2008).)

The figure makes it clear that the effect of disagreement is make consumption more volatile than without disagreement (when all investors consume the common period 2 income): a pessimist will tilt his portfolio towards assets that he thinks will pay off—so it becomes less diversified. The reason is that he thinks he gets a good deal on those assets,

[6] More generally, with logarithmic utility functions (and the same C_{i1} choice by all investors), (5) shows that asset prices depend on average beliefs (pdf values).

since optimists are willing to supply (sell) them. This extra volatility induces a high risk premium for all assets that are positively correlated with consumption. This is the basic mechanism for why disagreement matters for asset pricing.

The asset prices and returns in (8)–(11) also hold exactly when all investors agree (set $\delta = 0$)—since they then all make the same choices. In this case, the relevant uncertainty is $\omega^2 = \sigma^2$. The approximate results can thus be seen as an approximation around either logarithmic utility or no disagreement.

3.3 Numerically Calculated Asset Prices

I solve the model numerically by discretising the continuous state space (using 275 states) and the continuos cross-section of investors (using 51 investors). In the base case, I use the following parameter values: $\beta = 1$, $\mu = \sigma = \delta = 0.02$. This corresponds to an expected output growth of 2% with an individual standard deviation of 2% (so a 90% confidence band is -1.28 to 5.28) and a disagreement of the same size. The volatility numbers are plausible for 3-year GDP growth according the survey evidence below (the mean growth rate does not matter for the risk premia). (See Appendix A for details.)

Figure 1.c illustrates that the numerically calculated asset prices (for log utility) are very similar to the exact results. Indeed, the equity risk premium from the numerical calculation is virtually identical (a ratio of 0.997–0.999) to the exact result even if the uncertainty or disagreement is scaled down or up by a factor of ten (compared to the numbers used in Figure 1). Similarly, the numerical results obtained at higher risk aversions (for instance, $\gamma = 5$) but no disagreement are equally close to the exact result (details are not reported). This suggests that the numerical results are accurate.

Figures 2.a–b show numerical and approximate equity premia (from (10)) with a risk aversion (γ) of 3. Subfigure a shows how the risk premium changes as the individual uncertainty (σ^2) is scaled up or down compared to a base case (the base case has $\sigma = \delta = 0.02$), while subfigure b shows the effect of scaling the base case disagreement (δ^2). It seems as if the approximate results are very similar to the numerical results. In particular, it is clear that changing individual uncertainty has a much larger effect than changing disagreement—and that the difference is larger with a high risk aversion.

Similarly, *Figures 2.c–d* show the "implied variances" from call options obtained by "backing out" the unknown variance from the Black-Scholes formula. The calculations use options which are "at the money," that is, when the strike price equals the forward

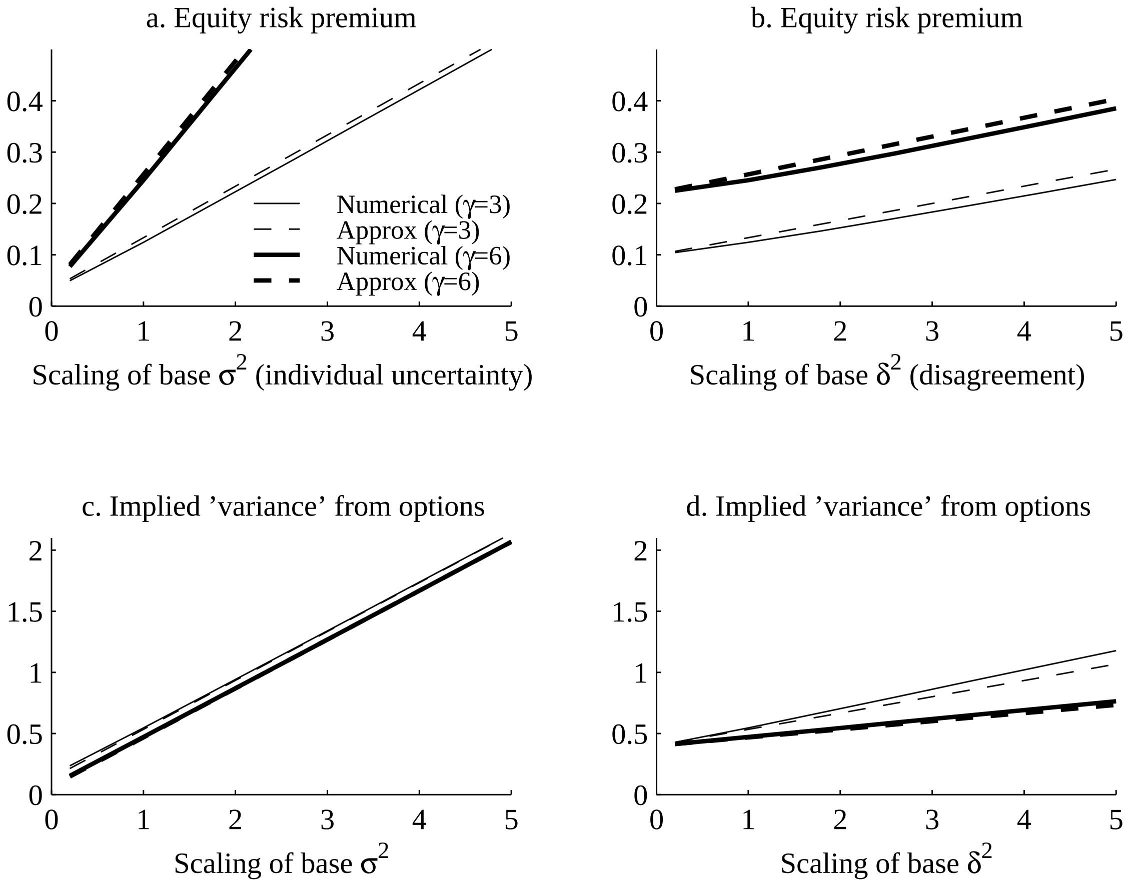

Figure 2: Equity risk premia and option implied volatilities for $\gamma = 3$. The base values are $\sigma^2 = \delta^2 = 0.02^2$ (and all figures use $\mu = 0.02$). σ^2 is scaled (up and down) in subfigures a and c, while δ^2 is scaled in subfigures b and d. The implied variance from an option is calculated at a strike price equal to the forward price of the consumption claim (which equals the price of a consumption claim times $\exp(r_f)$).

price of the consumption claim. (Unreported results show that other strike prices give very similar results.) Once again, the numerical and approximate results are very similar—and increasing individual uncertainty has a larger effect than increasing disagreement, especially when the risk aversion is high.

To get an intuition for why the approximate results are so close to the (almost exact) numerical results, *Figure 3.a* shows the numerical solution for the period 1 consumption choice (with $\gamma = 3$). Both optimists and pessimists consume more than the median investor (compare with the case of logarithmic utility, when all investors choose the same period 1 consumption). The reason is that both groups believe they face really good

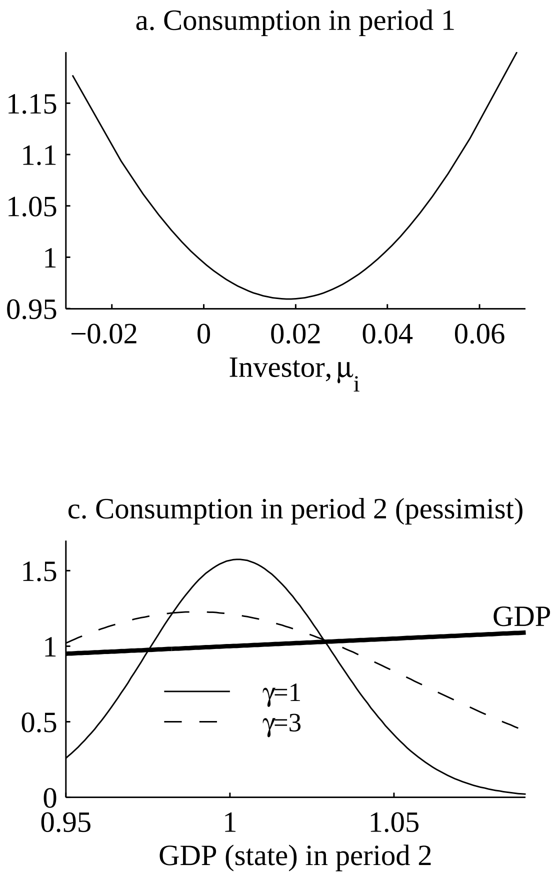

Figure 3: Consumption choice and cross-sectional distribution, $\gamma = 3$. This figure shows results based on $\mu = \sigma = \delta = 0.02$.

deals on the asset markets. Pessimists can buy assets that pay off in poor states (which they think will happen) really cheap—and analogously for optimists. This means that the economic weight, $C_{i1}g_i$ in (5), of a pessimist is higher than the relative frequency, g_i, as illustrated in *Figure 3.b*. Since the pessimists demand assets that pay off in poor states, this will increase the prices of those assets. Similarly, prices for really good states will also be driven up (because of the optimists). This suggests that the approximate AD prices (8) underestimate prices of extreme states and overestimate prices of median states—which is verified by (unreported) numerical calculations. However, the effect on the equity premium and the option prices is small—for two reasons. First, most investors choose, after all, fairly similar period 1 consumptions. Second, the consumption claim is essentially an average of all the AD assets, so the under- and overpricing more or less cancel. Unreported numerical results suggest that it takes a combination of a really low risk aversion ($\gamma < 1/2$) *and* a very high disagreement relative to individual uncertainty (at least a factor 5) to make the numerical results deviate noticeably from the approximate expressions.

With high risk aversion, investors do not dare to concentrate the consumption plan (to a few states) as much as with logarithmic utility. This is illustrated in *Figure 3.c* which shows the consumption plans for logarithmic utility and for $\gamma = 3$ (period 2 income is also indicated). This effectively reduces the volatility of individual consumption, which counterbalances the higher risk premium that naturally follows from a higher risk aversion. This is the basic reason for why disagreement is relatively less important when risk aversion is high. We can therefore think of the equity risk premium (10), $\gamma(\sigma^2 + \delta^2/\gamma)$ plus a Jensen's inequality term, as follows: the premium equals risk aversion times the effective risk (term in parentheses), but effective risk is decreased when risk aversion is high (hence the $1/\gamma$ term).

4 Survey Evidence on Individual Uncertainty and Disagreement

The Survey of Professional Forecasters includes evidence on professional forecasters' beliefs about GDP. It asks for probability distributions (histograms) of GDP growth for the current and the next calendar years, and also about the point forecasts (no histograms) about the average growth over the next 10 years (see Croushore (1993) for details).

I fit distributions to the individual histograms as follows. In the uncommon case where only one bin is used (that is, the respondent puts 100% of the probability on one bin), I assume a triangular distribution within that bin. If two bins or more are used, then a normal distribution is fitted (a mean and a variance). The fitting is done by minimising the sum of the squared deviations of the theoretical from the observed probabilities (see Giordani and Söderlind (2003)).

This estimation gives a cross-section of individual point forecasts (means) and variances in each time period (the first quarter of each year 1992–2007). Since there are occasional strange survey answers (typos?), I apply robust estimators to the cross-section in each period: "average" individual uncertainty is estimated by the median variance (from the histograms) and disagreement is estimated by the square of the interquartile range (divided by 1.35^2 to be comparable to a variance).

Table 1 shows time-averages of these results. Individual uncertainty is more than twice as large as the disagreement for both the first and the second year. In addition, the disagreement about average 10-year growth is just half of the disagreement about 1-year growth. This suggests that it is plausible to assume that individual uncertainty dominates disagreement also for longer horizons (unfortunately, there is no data on 10-year individual uncertainty).

	Median individual variance, σ^2	Cross-sectional variance, δ^2
Current year	0.41	0.19
Next year	0.62	0.24
Next 10 years (average)		0.11

Table 1: Individual uncertainty and disagreement about GDP growth, Survey of Professional Forecasters. The table shows time-averages from Q1 surveys 1992–2007 by the Survey of Professional Forecasters. The median individual variance (for a given period) is estimated from the fitted individual variances. The cross-sectional variance (for a given period) is estimated by the square of the inter-quartile range of the individual point estimates, divided by 1.35^2. The latter is a robust estimator of the variance and coincides with a traditional estimator in case the data is normally distributed.

5 Summary

Numerical and (sometimes approximate) analytical solutions of a two-period endowment model suggest that the uncertainty of a "representative investor" is $\sigma^2 + \delta^2/\gamma$, where σ^2 denotes individual uncertainty, δ^2 is a measure of disagreement (cross-sectional variance of investors' point estimates) and γ is the coefficient of relative risk aversion. Therefore, the effect of disagreement on asset prices is small (compared to the effect of individual uncertainty) when risk aversion is high—and when individual uncertainty and disagreement are of the same order of magnitude.

Both requirements are likely to hold. First, the empirical literature on consumption-based asset pricing suggests that risk aversion is high. Second, results from the survey data from the Survey of Professional Forecasters indicate that individual uncertainty is actually larger than disagreement.

To arrive a clear results, this model is kept simple. It is therefor not a suitable framework for serious econometric estimation, or for generating general conclusions. It only shows that it is hard to assign disagreement a major role in a stylised consumption based asset pricing model.

A Numerical Solution of the Model

This appendix summarises the numerical solution of the model.

The (continuous) cross-section of investors is approximated by m investors: the means $(\mu_i, i = 1, ..., m)$ are the $i/(m+1)$ percentiles of the $N(\mu, \delta^2)$ distribution. The (continuous) state space is approximated by n states, equally spaced (in logs) from percentile θ of the most pessimistic investor to percentile $1-\theta$ of the most optimistic investor. The probability of state s for investor i is calculated as the integral of a $N(\mu_i, \sigma^2)$ pdf (denoted $f_i(s)$) over the interval $(s-\lambda, s+\lambda)$, where λ is halfway to the next state.

The results are obtained by solving the following system of equations (in terms of C_{i1})

$$Y_{i1} - C_{i1} - \sum_{s=1}^{m} p(s) B_i(s) = 0 \text{ for } i = 1, ..., m, \text{ where}$$

$$\begin{aligned} p(s) &= \left(\sum_{i=1}^{m} f_i(s)^{1/\gamma} C_{i1}/m\right)^{\gamma} / s^{\gamma} \text{ for } s = 1, ..., n, \\ C_{i2}(s) &= f_i(s)^{1/\gamma} p(s)^{-1/\gamma} C_{i1}, \\ B_i(s) &= C_{i2}(s) - Y_{i2}(s). \end{aligned}$$

In the calculations $n = 275$, $m = 51$ and $\theta = 1e-6$.

B Derivations of Asset Prices

When C_{i1} is the same for all investors, (or more generally, when $f_i(s)^{1/\gamma}$ and C_{i1} are uncorrelated), then (5) can be written

$$p(s) = \beta \left(\int_i C_{i1} g_i di\right)^{\gamma} \left(\int_i f_i(s)^{1/\gamma} g_i di\right)^{\gamma} / s^{\gamma} = \beta \left(\int_i f_i(s)^{1/\gamma} g_i di\right)^{\gamma} / s^{\gamma},$$

since $\int_i C_{i1} g_i di$ equals average income in period 1, which is normalised to unity. Straightforward integration gives (8). (See also Giordani and Söderlind (2005) for details on the integration.)

Using the approximate asset prices (8) and integrating, $p_r = \int_s p(s)ds$ and $p_c = \int_s p(s)sds$ gives

$$p_r = \beta(\omega/\sigma)^{1-\gamma} \exp\left(-\gamma\mu + \gamma^2\omega^2/2\right),$$
$$p_c = \beta(\omega/\sigma)^{1-\gamma} \exp((1-\gamma)\mu + (1-\gamma)^2\omega^2/2).$$

The log riskfree rate, $\ln(1/p_r)$, is then as in (9) and the excess log return on the consumption claim, $\ln(Y_2/p_c) - \ln(1/p_r)$, is

$$r_c - r_f = -\mu - (1-2\gamma)\omega^2/2 + \ln Y_2.$$

Take the average expectations over $\ln Y_2$, $\int_i \mu_i g_i di = \mu$, to get (10).

To price a call option on the consumption claim with strike price K, use the approximate asset prices (8) and integrate, $\int_{s=K}^{\infty} p(s)(s-K)ds$ to get

$$(\omega/\sigma)^{1-\gamma} \exp((1-\gamma)\mu + (1-\gamma)^2\omega^2/2)\frac{1}{2}\text{erfc}\left(\frac{(\gamma-1)\omega^2 - \mu + \ln K}{\sqrt{2}\omega}\right) -$$
$$(\omega/\sigma)^{1-\gamma} \exp(-\gamma\mu + \gamma^2\omega^2/2)K\frac{1}{2}\text{erfc}\left(\frac{\gamma\omega^2 - \mu + \ln K}{\sqrt{2}\omega}\right),$$

where erfc is the complementary error function. This can be simplified by noticing a few things. First, the exponential part of the first term equals p_c and the exponential part of the second term equals p_r (see above). Second, $\text{erfc}(b/\sqrt{2})/2 = \Phi(-b)$, where $\Phi()$ is the cumulative distribution function of a standard normal distribution. Third, the negative of the argument of the first erfc function can be rewritten as $\ln(p_c/p_r) + \omega^2/2$ (use the definitions of p_c and p_r from above). Fourth, the negative of the argument of the second erfc function equals the negative of the argument of the first erfc function minus ω. Together, these simplifications give the call option price

$$p_c\Phi(d_1) - p_r K\Phi(d_1 - \omega), \text{ where } d_1 = \frac{\ln p_c/K - \ln p_r + \omega^2/2}{\omega},$$

which is the Black-Scholes formula (recalling that $p_r = \exp(-r_f)$), where ω^2 plays the role of the volatility (variance).

The formulas for the approximate asset prices apply also to the cases of logarithmic utility ($\gamma = 1$) and no disagreement ($\delta = 0$, so $\omega = \sigma$)—but then exactly (since C_{i1} is indeed the same for all investors).

With logarithmic utility, the asset prices are

$$p(s) = (\beta/s)\varphi(s; \mu, \sigma^2 + \delta^2).$$

Combine with (3) (with $\gamma = 1$) and $f_i(s) = \varphi(s; \mu_i, \sigma^2)$ as well as $C_{i1} = 1$ to get period 2 consumption

$$C_{i2}(s) = s\frac{\varphi(s; \mu_i, \sigma^2)}{\varphi(s; \mu, \sigma^2 + \delta^2)}.$$

With no disagreement, the asset prices are

$$p(s) = (\beta/s^\gamma)\varphi(s; \mu, \sigma^2).$$

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