A Satisficing Alternative to Prospect Theory

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Abstract

Motivated by extensive research suggesting that aspiration levels play a central role in the decision making of many individuals, we introduce a target-based model of risky choice. A key aspect of the model is the decision maker's attitude towards risk: when prospects appear favorable relative to the target, the decision maker behaves in a risk averse way. In sufficiently dire circumstances, however, prospects may be in danger of failing to meet the target, and the decision maker can display “risk-seeking” behavior. By incorporating both targets and such a switch between risk attitudes, the approach can be viewed as a hybrid model, capturing in spirit the celebrated ideas of both satisficing and prospect theory. The model is simple to motivate and applicable in settings where target attainment is of central concern to the decision maker. We show that such an approach is a natural dual to one based on risk measures. We establish structural properties, such as stochastic dominance, and show that the model has a general interpretation in terms of an index ranking prospects in the face of distributional ambiguity. Finally, though the model is simple and not designed with a specific intent to address any puzzles of decision theory, an interesting byproduct is that it can accommodate a number of “paradoxes,” both neoclassical and classical (e.g., Allais and Ellsberg).

Keywords
satisficing; aspiration levels; targets; prospect theory; reflection effect; risk measures; coherent risk measures; convex risk measures; portfolio optimization.

JEL Classification
D81, G11.
1 Introduction

The notion of an aspiration level rests at the core of Simon’s [37] concept of bounded rationality. A central theme of this paradigm is that, due to limited cognitive resources and incomplete information, real-world decision makers may plausibly follow heuristics in the face of risky choice: namely, satisficing behavior, in which the decision maker accepts the first encountered alternative that meets a sufficiently high aspiration level, may prevail.

It is not difficult to imagine situations in which decision makers possess an aspiration level, or “target.” For instance, the compensation of portfolio managers is closely linked to performance relative to a benchmark. Individuals often choose to invest personal savings for the purpose of meeting costly, downstream goals, such as retirement or their children’s education. Consumers looking to make significant purchases or sales may have a reservation price in mind. In such situations, a target payoff or price seems fairly natural, and it seems plausible that the realized performance relative to this target will be a key standard by which the decision maker ultimately measures success. A number of descriptive studies confirm this intuition and find, for real-world managers, that aspiration levels are a key driver of their decision-making (e.g., Lanzilloti [24], Mao [28], Payne et al. [31, 32]). More recently, Payne [30] provides extensive experimental evidence that decision makers are highly sensitive to perturbations in the probability of a loss or a gain. Diecidue and van de Ven [10] argue compellingly for the importance of aspiration levels and develop a corresponding model that they show can be represented by discontinuous utility functions.

At the same time, a cornerstone of most theories of risky choice is the idea of risk aversion. Whether any deviations from risk aversion should unilaterally be deemed “irrational” is the subject of considerable dispute. This debate traces back at least to Friedman and Savage [16] and Markowitz [29], who argue that it is not unreasonable at times for preferences to incorporate some degree of risk seeking behavior. There is also considerable descriptive evidence that real-world decision makers are risk seeking, particularly for gambles involving losses (e.g., Fishburn and Kochenberger [13], Hershey and Schoemaker [20], Payne et al. [31], and many others). This idea has then been built into various models of choice, most famously in the prospect theory of Kahneman and Tversky [21].

The goal of this paper is to develop a model of choice with the notion of a target goal as its key primitive. The model also allows for decision makers, in sufficiently dire circumstances, to exhibit risk seeking tendencies, thereby capturing the central theme of prospect theory. Because of the link to these two, core ideas, we call the resulting model of choice prospective satisficing preferences (PSP).

Our framework is based on properties that we feel are relatively simple and straightforward to motivate in settings where a target goal is the key focus of the decision maker. The basic properties of our PSP choice model can be succinctly described as follows:

(a) *Satisficing*: a position that can surely beat the target will be maximally preferred; conversely, a position that definitely cannot attain the target will be least preferred.

(b) *Monotonicity*: if one position always outperforms another (in any state), then the outperforming position is always preferred among the two.
(c) *Diversification behavior*: diversification is preferred among positions that are somehow “secure” relative to attaining the target; concentration is preferred among positions that are somehow “vulnerable” relative to attaining the target.

(a) captures the spirit of satisficing: decision makers following this model have a target goal in mind and are fully content to select a position that guarantees that they will attain this target. (b) simply means such decision makers do not prefer less to more. Finally, (c) suggests that such a decision maker will prefer to mix among positions to make the net stake more secured in terms of target attainment, but only when such positions are already relatively safe bets; if the available positions are in danger of missing the target, rather than mixing between these unfavorable prospects, the decision maker prefers to “roll the dice” and select a single position that will provide the best hopes of attaining the target. Risk seeking behavior of this flavor does not seem outrageous when achieving a target is the decision maker’s focal point. This is, of course, a rough description of the model, and we will motivate and discuss it in more detail later.

The PSP model has ties to the approach of *satisficing measures* recently proposed by Brown and Sim [5]; specifically, the model of choice induced by (a) and (b) above can be shown to be equivalent to choice with a satisficing measure in [5]. Related performance measures have also been developed. Cherny and Madan [6] develop what they call an “acceptability index” (geared towards absolute performance measurement, i.e., a target of zero) that fall into the class of coherent satisficing measures of [5]. Aumann and Serrano [3] have axiomatized an “index of riskiness” that is the reciprocal of the entropic satisficing measure of [5]. Recently, Drapeau and Kupper [11] axiomatized risk preferences that satisfy (b) and quasi-concavity, which results in a model that includes the quasi-concave satisficing measures of [5].

Property (c), however, results in a model distinct from those above, which are models that favor diversification. The allowance of some amount of risk seeking behavior in the PSP model here is a nontrivial deviation that results in a significantly different representation. Moreover, this difference has important implications from a choice perspective. For instance, under fairly mild assumptions, using a satisficing measure from [5], a decision maker would be unable to distinguish among a set of prospects that all have negative expected values (with respect to the target). Such a model is therefore not well-suited to handle choice problems that involve a selection among unfavorable positions. By contrast, in the PSP model, one can distinguish among such positions.

A further difference from [5] is that we explore some of the descriptive relevance of PSPs. In particular, we find that the model can resolve the paradoxes of Allais [1] over a fairly wide range of targets. In addition, because PSPs do not require a particular probability measure on the outcome space, they can therefore naturally incorporate ambiguity, thereby addressing the criticism of Ellsberg [12]. We also find that this model can accommodate some more recent, “neoclassical” puzzles of decision theory. We would like to state upfront that we do not motivate the model with a particular descriptive intent in mind; our goal is simply to provide a target-based model of choice that is relatively easy to describe and justify. The descriptive relevance of the model is purely a byproduct and is, as we see it, somewhat surprising as a result.

At the outset, we mention that this model is simply a proposal for risky choice that we feel is reasonably well-motivated in settings where a target goal is the primary focus of the decision maker. Like any model of choice one can surely construct example sets of risky prospects for which rigidly
adhering to PSP can lead to choice behavior that may seem puzzling. Expected utility theory is of course no exception here, as many have called into question the independence axiom or the inability of EUT to handle ambiguity in subjective probabilities (see, e.g., Machina [26], Quiggin [33] and Yaari [41], and Gilboa and Schmeidler [18], among numerous other modifications of expected utility theory).

Nonetheless, we feel that the model possesses a number of features that are practically useful and theoretically desirable. First, the key parameter for practical use of the model is the target goal. This is in contrast to a model of choice, like one based on expected utility values or risk measures, that requires specification of some sort of tolerance parameter. While it is true that one can attempt to elicit utility functions or their parameters (see, e.g., Wakker and Deneffe [39] for a discussion of this process in the face of ambiguity), it seems hard to argue that such elicitation is more natural than specifying targets. For instance, it is easy to imagine that many practicing managers may have a fairly good idea of reasonable target performance they would like to attain. On the other hand, it is more difficult to envision such a manager firmly believing that they should weigh monetary outcomes according to, say, a power function, then performing reference gamble experiments to determine an appropriate coefficient of relative risk aversion. In fact, it seems likely that many individuals may not even be able to assign a real meaning to such a coefficient; on the other hand, a target goal is a concrete entity that anyone can understand.

The PSP model, in contrast to many choice models, does not attempt to directly assign a value to each possible monetary outcome, and we caution the reader upfront to take great care in making direct comparisons to such models. We will prove a general representation result that states that choice under PSP is, in fact, equivalent to ranking prospects according to an index level. This index level is the maximum level at which a corresponding risk measure is “acceptable.” Over positions that are “secure” relative to the target and for which diversification is preferred, the risk measures in the representation are shown to be convex risk measures, developed by Föllmer and Schied [14] as a generalization of the coherent risk measures introduced by Artzner et al. [2]. Over positions that are “vulnerable” relative to the target, on the other hand, the representing risk measures will be concave.

This representation result shows that choice under PSP is dual to choice under risk measures. An important consequence of such a representation is that it allows us not only to describe and implement PSP choice in terms of already understood objects (for example, the choice function can be induced by conditional value-at-risk or CARA utility; as another example, under some assumptions on the space of random variables, Sharpe ratio is a special case of our model), but it also allows us to leverage known properties of such risk measures. For instance, under mild assumptions we can show that the PSP model obeys first-order stochastic dominance everywhere and second-order stochastic dominance over secured positions. We note that this would not be true if one were to simply rank positions according to their probability of hitting the target.

Finally, an appealing feature of the PSP model is that it does not require specification of a probability distribution. If one has probabilities, they can of course be used, but such a description is not necessary for the model. In fact, we will provide a generic interpretation of the model in terms of a risky position’s expected payoff in the face of distributional ambiguity. This ambiguity interpretation is a consequence of the dual representation and is not obvious from the PSP properties themselves; this brings us, interestingly enough, full-circle back to Simon’s [37] notion of bounded rationality, which posits that
probabilities are typically not fully precise quantities.

Before moving forward, we would like to state that the terms “satisficing” and “prospective” here should not be taken fully literally, but are instead capturing these ideas in spirit. Simon [37] certainly had a very specific definition of satisficing; our model does not state that decision makers should satisfice in the strictest sense of the word, as in waiting for the first opportunity that exceeds the target. In fact, in the context of risky choice, this is not even a meaningful mode of behavior, as payoffs are uncertain. Moreover, we can talk about optimizing a satisficing measure, a phrase that may seem like an oxymoron to purists. The idea is mainly that we are capturing some of the essence of the satisficing idea via target goals as the centerpiece of the model. A similar story holds with prospect theory; the PSP model is quite different than the one developed by Kahneman and Tversky [21] and one cannot directly map between the two. Nonetheless, our model utilizes a key component of their theory: namely, risk attitudes that somehow vary according to performance relative to a target.

The remainder of the paper is as follows. In Section 2, we first introduce the model and then give the main theoretical result of the paper, which is the representation theorem in terms of risk measure families. We then present some examples of PSPs and provide mild conditions under which stochastic dominance orders hold. In Section 3, we show how the model can resolve some of the inconsistencies of expected utility theory stemming from observed violations of the independence axiom. Section 4 discusses how to use PSPs in settings of ambiguity, e.g., situations where the Ellsberg paradox emerges. Section 5 discusses optimization with our model and provides an example application to portfolio choice. Finally, Section 6 concludes the paper. All proofs are in the appendix.

2 Prospective satisficing and its representation

In this section, we introduce the model and show its representation in terms of a classical notion of “risk measures.”

Uncertainty is given by a state-space \( \Omega \) and a set (sigma-algebra) \( \mathcal{F} \) of events \( A \subseteq \Omega \). We denote by \( \mathcal{V} \) the set of feasible prospects \( V: \Omega \to \mathbb{R}; V \in \mathcal{V} \) can be viewed as the random payoff tomorrow of a particular alternative chosen today. Note that we do not assume a specific probability measure on \( \Omega \).

We consider the situation of a decision maker who wants to choose a prospect from \( \mathcal{V} \). The decision maker has an aspiration level \( \tau \), which is another prospect on \( \Omega \), not necessarily in \( \mathcal{V} \). If \( \tau \) is constant, it can simply be thought of as a fixed target payoff; otherwise, it can be viewed as a reference prospect or benchmark, also random, that the decision maker wishes to outperform.

We assume that the target is specified exogenously (i.e., it is an input from the decision maker). We thus consider only payoffs relative to this target; in particular, we define the set of target premia to be

\[
\mathcal{X} = \{ V - \tau : V \in \mathcal{V} \},
\]

and we will henceforth suppress the notation of the target. \( X(\omega) \geq 0 \) means the target has been achieved (or exceeded) in state \( \omega \); \( X(\omega) < 0 \) means the target has not been attained in state \( \omega \). The notation \( X \geq Y \) denotes state-wise dominance, i.e., \( X(\omega) \geq Y(\omega) \) for all \( \omega \in \Omega \).

It is worth emphasizing that the space of target premia is a set of random variables, not a set of lotteries. For the latter, the payoffs in a given state would typically be fixed across the set, and
one would look at varying the probability distribution. With a set of random variables, however, this restriction need not be imposed. This seems more natural in many settings; for instance, in a portfolio choice context, the states may correspond to different return outcomes, and the set of random variables may correspond to different allocation choices. In this setting, different positions would pay differently in the same state, and therefore describing this as a set of lotteries seems rather unnatural.

We model decision maker’s preferences using a preference relation $\succeq$ on $X$. For target premia $X, Y \in X$, the decision maker prefers $X$ to $Y$ if and only if $X \succeq Y$. As usual, $\succ$ and $\sim$ are defined by $[X \succ Y \iff (X \succeq Y) \text{ and } \neg(Y \succeq X)]$ and $[X \sim Y \iff (X \succeq Y) \text{ and } (Y \succeq X)]$.

We begin with the standard assumption that the preference order is complete, reflexive, and transitive, in addition to some technical conditions.

**Property 1** (Weak order and upper semi-continuity). Let $\succeq$ be a weak order on $X$ that satisfies:

(i) For all $X \in X$, the set $\{Y \in X : Y \succeq X\}$ is closed in $X$ (upper semi-continuity).

(ii) There exists $Z \subset X$ that is order-embeddable\(^1\) into $[0, 1]$ such that for all $X, Y \in X$ with $X \succ Y$, there exists $Z \in Z$ such that $X \succeq Z \succeq Y$.

The assumption of a weak order is standard. Properties (i) and (ii) are technical conditions needed to obtain a functional representation of $\succeq$, which we will do in Subsection 2.1. For the purpose of this paper, the classical continuity assumption as defined by Debreu [8] is too restrictive, since it excludes preference relations in the spirit of Simon [37] (e.g., $X \succeq Y$ if and only if $\rho(X) \geq \rho(Y)$, where $\rho : X \rightarrow \mathbb{R}$ satisfies $\rho(X) > 0$ on $\{X \geq 0\}$ (satisfactory payoffs) and $\rho(X) < 0$ on $\{X < 0\}$ (unsatisfactory payoffs)).

Following a generalization of Rader [34] we thus impose upper semi-continuity (Property (i)). This is obviously weaker than continuity and allows preference relations that will have a satisficing flavor.\(^2\)

We then require the following.

**Property 2** (Monotonicity). For all $X, Y \in X$, if $X \geq Y$ then $X \succeq Y$.

Monotonicity says that a premium that dominates another premium state-wise is preferred among the two. This is a classical assumption on preferences, and it says that decision makers do not prefer less to more.

In the spirit of Simon [37], we next have the following.

**Property 3** (Satisficing behavior).

(i) Attainment content: If $X \geq 0$, then $X \succeq Y$ for all $Y \in X$.

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\(^1\)Z is order-embeddable into $[0, 1]$ when there exists a order-preserving function $f : (Z, \succeq) \rightarrow [0, 1]$.

\(^2\)Upper semi-continuity is not enough for a functional representation of $\succeq$. As in the classical result of Debreu [8], we need to impose some countability conditions on the topological space. Rader [34] assumes that the topological space is second countable. This, in addition to upper semi-continuity, is sufficient to derive a functional representation of $\succeq$, but it is not necessary: it can be weakened to hold on any topological space if an additional condition holds. Namely, $\succeq$ must possess a functional representation on a subset $Z$ of $X$ and for any jump $(X, Y)$ in $X$ there must be an element $Z \in Z$ with $X \geq Z \geq Y$ (a jump in $X$ is a pair of elements $(X, Y) \in X \times X$, $X \succ Y$, such that $\{Z \in X : X \succ Z, Z \succ Y\} = \emptyset$). This is Property (ii). Note that since $Z$ can be assumed to be countable (see Bosi and Mehta [4]), Property (ii) implies that $X$ only contains countable many jumps.
Non-attainment apathy: If $X < 0$, then $Y \succeq X$ for all $Y \in X$.

Satisficing behavior relates to the properties of the satisficing measures of Brown and Sim [5]. Satisficing measures are functions $\rho : \mathcal{X} \rightarrow [0, 1]$ that will lead to choosing a position $X$ over $Y$ if and only if $\rho(X) \geq \rho(Y)$. The key properties of satisficing measures are as follows. First, monotonicity: if $X \geq Y$, then $\rho(X) \geq \rho(Y)$. Next, attainment content: namely, if a position always beats the target, (i.e., $X \geq 0$), then we are fully satisfied with it (i.e., $\rho(X) = 1$). Conversely, non-attainment apathy: if a position never beats the target (i.e., $X < 0$), then we are fully unsatisfied with it (i.e., $\rho(X) = 0$).

Property 3 is the analog for preference relations of these two latter properties of satisficing measures. This property captures the essence of a target-driven decision maker: namely, achieving this aspiration level is paramount, and prospects that always (never) do so should be most (least) highly valued. This squares with a wide body of empirical evidence of how real-world managers operate; Mao [28], for instance, concludes after interviewing many executives that “risk is primarily considered to be the prospect of not meeting some target rate of return.” Diecidue and van de Ven [10] make a convincing case on this point and provide many more references based on corroborating empirical evidence. Managers are certainly not alone in this behavior: Payne [30] recently showed that many decision makers would be willing to accept a decrease in a gamble’s expected value in order to reduce just the probability of a loss.

One may observe that if every position in $\mathcal{X}$ is always above (or always below) the target in every state, then such preferences will be indifferent to every choice in $\mathcal{X}$, and one may question the usefulness of the model in such situations. We see two primary counterarguments to this critique. First, if a decision maker truly does possess a target that is so low that every available position beats it in every state of the world, then such an unambitious decision maker may very well be willing to simply accept any of them.

Alternatively, and perhaps more realistically, it may often be the case that targets are tied to the available opportunity set: the decision maker would be unlikely choose an extreme target that is overly conservative (or unreasonably ambitious). In a portfolio choice context, for instance, it seems somewhat implausible for the investor to choose a target that is so high that there is no state in which it can ever be attained for any possible investment choice. On the flip side, if the target is so low that every possible position always beats it in every state, then one may reasonably believe such an unmotivated investor is perfectly happy with a simple, risk-free investment. So, while it is true that the above preferences cannot distinguish across positions in these cases, such circumstances demand seemingly extreme targets, and, in these situations, the choice problem itself seems rather uninteresting in the first place. Of course, the issue of how individuals form targets is an interesting one; again, here we are taking the target to be exogenously specified by the decision maker.

On a related note, we concede that Property 3 makes our model ill-suited to distinguish among a set of purely deterministic gambles. By monotonicity and Property 3, decision makers would indeed be indifferent to all sure bets above (or below) the target. We would argue, however, that such decision settings are rare in the real world and not of much interest. Uncertainty plays a key role in real-world

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3Gilboa and Schmeidler [19] provide a model of aspiration level adjustment over time and argue that a “realistic” model of such adjustments should relate aspiration levels to past performance in similar situations. Though the overall model is very different from ours, their perspective seems to align well with our above appeal to “reasonable” target levels.
decisions, and where our model has the most discriminatory power are in challenging situations when the relevant prospects have some chances of falling above and below the target.

A simple example of a ranking function that would satisfy the above preferences is $\rho(X) = \mathbb{P}\{X \geq 0\}$. The difficulty with using probability of beating the target as a basis for choice is that in general it will not favor diversification, and at least some degree of diversification-favoring or "risk aversion" behavior is generally considered to be a desirable feature of most models of risky choice. Using the probability of beating the target as a guide for portfolio choices, for example, optimal allocations may often be those that are highly invested in a small number of assets if not a single asset. Furthermore, the probability of beating a target is a very crude tool for decision-making: it is easy to construct examples of two target premia with one first-order stochastically dominating the other but for which both have the same probability of beating the target. A related downside is that, computationally, maximizing the probability of beating the target is, in general, a very difficult optimization problem.

Brown and Sim [5] show how to induce a desire for diversified positions in the satisficing measure by imposing that $\rho$ be a quasi-concave function (i.e., $\rho(\lambda X + (1 - \lambda)Y) \geq \min\{\rho(X), \rho(Y)\}$). This seems desirable for positions for which we are relatively confident that the target will be attained; on the other hand, it seems implausible that investors would always want to diversify among positions that are in danger of not attaining the target.

As a simple illustration of this, consider a case with $|\Omega| = 2$, and consider the two positions $X = \{0, -1\}$, $Y = \{-1, 0\}$. Both of these positions may attain the target, but notice that any convex combination $\lambda X + (1 - \lambda)Y$ results in the position $Z = \{-1 - \lambda, -\lambda\}$, which never attains the target. If attaining the target is truly the investor’s goal, then ranking every such $Z$ as not worse than both $X$ and $Y$, as a quasi-concave satisficing measure would, seems problematic. Of course, if the positions are reversed in sign, then it certainly does seem to make sense to value diversification; the position $Z$ now is $\{1 - \lambda, \lambda\}$ and always attains the target, so it should not be worse than both $X = \{0, 1\}$ and $Y = \{1, 0\}$.

Another example, which relates to the modern portfolio theory of Markowitz [29], is the following. Consider an investor who wishes to maximize the Sharpe ratio of a portfolio relative to a target rate of return $\tau$ (this is a classical approach in portfolio theory and if returns are normally distributed, this is equivalent to the problem of maximizing the probability of beating the target). Denote the mean and covariance of the returns by $\mu$ and $\Sigma$, respectively. The investor’s problem can be expressed as

$$\max \left\{ \frac{\mu'w - \tau}{\sqrt{w'\Sigma w}} : w \in \mathcal{W} \right\},$$

where $\mathcal{W}$ is a feasible set of portfolio choices. The properties of the objective function depend on the size of $\tau$: it is a quasi-concave function over the set of $w \in \mathcal{W}$ with mean return no smaller than $\tau$ (and hence diversification will be favored here). On the other hand, if there are no such $w \in \mathcal{W}$, then the objective is actually quasi-convex and concentration is preferred. Thus, we find even fairly classical settings in which diversification is not always desirable.

These examples suggest that a target-oriented model of choice may need to be more discriminating in whether it values diversification or not. This is somehow intuitive: if circumstances are ominous and attaining a target is of high priority, a decision maker may not want to rest on their laurels and take very conservative positions. Moreover, there is a rather large body of empirical evidence that suggests that
real-world decision makers are risk averse for gains and risk seeking for losses; therefore, incorporating some risk seeking behavior potentially enhances the descriptive relevance of the model.

This motivates the following:

**Property 4** (Prospective behavior). *There exists a partition of \( \mathcal{X} \) into three disjoint subsets \( \mathcal{X}^{++} \), \( \mathcal{X}^{--} \) and \( \mathcal{X}_0 \), termed the secured, vulnerable and neutral sets of target premia, respectively, such that for all \( X_- \in \mathcal{X}^{--} \), \( X_1, X_2 \in \mathcal{X}_0 \), \( X_+ \in \mathcal{X}^{++} \), we have

\[
X_+ \succ X_1 \sim X_2 \succ X_-,
\]

and the following conditions hold:

(i) Diversifying (convex preferences) over secured target premia: *For all \( X, Y \in \mathcal{X}^{++} \), if \( X \succeq Z \), \( Y \succeq Z \) then

\[
\lambda X + (1 - \lambda) Y \succeq Z \quad \forall \lambda \in [0, 1].
\]

(ii) Concentration over vulnerable target premia: *For all \( X, Y \in \mathcal{X}^{--} \), if \( Z \succeq X \), \( Z \succeq Y \) then

\[
Z \succeq \lambda X + (1 - \lambda) Y \quad \forall \lambda \in [0, 1].
\]

This property states that decision makers can categorize target premia in \( \mathcal{X} \) into three sets depending on their abilities to achieve the target: among these three, secured target premia in \( \mathcal{X}^{++} \) are most desirable, vulnerable target premia in \( \mathcal{X}^{--} \) are least desirable, and the target premia in the neutral set \( \mathcal{X}_0 \) are in between and are not ranked (decision makers do not display a strict preference among any of them). The sets \( \mathcal{X}^{++} \) and \( \mathcal{X}^{--} \) can, loosely speaking, be interpreted as sets of prospects for which there are relatively good and relatively poor chances, respectively, of attaining the target. Properties (i) and (ii) above capture the notion of when decision makers should and should not gain from diversification, i.e., when convex preferences should be imposed. Property (i) states that diversification is beneficial on the set of secured target premia, while (ii) says that concentration should be preferred on the set of vulnerable target premia.

Note that we have placed no particular restrictions on the set of secured or vulnerable premia. All that we are imposing is that there exists such a partition and that diversification (concentration) is desirable for positions that are preferred (not preferred) over this partition. We will later show that when we have a probability distribution describing the world, under mild conditions, the set of secured (vulnerable) premia will essentially be those with positive (negative) expected values. When we do not assume the existence of a particular distribution but instead have a set of possible distributions, this interpretation will also extend in a natural way to the worst-case or best-case expected value over the set of distributions. This structure for the secured and vulnerable sets is purely a consequence of the full preference relation, however, and it is not assumed in Property 4 above.

**Definition 1.** A preference relation \( \succeq \) that satisfies Properties 1-4 is called a prospective satisficing preference relation (PSP).
2.1 Representation of PSPs

The first result that we report is that prospective satisficing preference relations possess a functional representation. This functional representation is a fairly transparent analog of the preference definition; we present it mainly because it is sometimes convenient (particularly in the proof of our main representation theorem to follow) to work with a choice functional rather than a preference relation.

**Proposition 1.** A preference relation $\succeq$ is a prospective satisficing preference relation if and only if there exists an upper semi-continuous function $\rho : X \to \mathbb{R} \cup \{-\infty, \infty\}$ such that

$$X \succeq Y \iff \rho(X) \geq \rho(Y).$$

Moreover, $\rho$ satisfies the following for all $X, Y \in X$:

(i) Monotonicity: If $X \geq Y$, then $\rho(X) \geq \rho(Y)$.

(ii) Satisficing behavior:

(a) Attainment content: If $X \geq 0$, then $\rho(X) = \infty$.

(b) Non-attainment apathy: If $X < 0$, then $\rho(X) = -\infty$.

(iii) Prospective behavior:

(a) Superiority of secured premia and inferiority of vulnerable premia:

$$\rho(X) =
\begin{cases} 
> 0 & \text{if } X \in X_{++}, \\
= 0 & \text{if } X \in X_0, \\
< 0 & \text{if } X \in X_{--}.
\end{cases}$$

(b) Quasi-concavity (diversifying) over secured target premia: For all $X, Y \in X_{++}, \lambda \in [0, 1]$:

$$\rho(\lambda X + (1-\lambda)Y) \geq \min\{\rho(X), \rho(Y)\}.$$

(c) Quasi-convexity (concentrating) over vulnerable target premia: For all $X, Y \in X_{--}, \lambda \in [0, 1]$:

$$\rho(\lambda X + (1-\lambda)Y) \leq \max\{\rho(X), \rho(Y)\}.$$

**Definition 2.** An upper semi-continuous function $\rho : X \to \mathbb{R} \cup \{-\infty, \infty\}$ that satisfies properties (i)-(iii) in Proposition 1 is called a prospective satisficing measure (PSM).

A few words about units are in order. First, the fact that we take $\rho$ to map to $\mathbb{R} \cup \{-\infty, \infty\}$ is primarily for convenience (any other continuous interval, such as $[0, 1]$, would work). We will show shortly that PSMs are essentially ranking indices; thus, one should resist the temptation to interpret $\rho(X)$ as a value representing some kind of subjective worth of the prospect $X$. A PSM value is a quantity used to make relative comparisons and therefore the “units” of a PSM are of little importance. As such, one must also be careful in making “strength of preference” interpretations with a PSM.
Inspection of Proposition 1 shows that given a PSM \( \rho \) we can characterize the sets of neutral, secured and vulnerable sets as follows:

\[
\begin{align*}
X_0 & = \{ X \in \mathcal{X} : \rho(X) = 0 \} \\
X_{++} & = \{ X \in \mathcal{X} : \rho(X) > 0 \} \\
X_{--} & = \{ X \in \mathcal{X} : \rho(X) < 0 \}.
\end{align*}
\] (2)

Though we have shown existence of a functional representation for PSPs, the properties of a PSM are fairly straightforward analogs of the properties for a PSP. Moreover, the properties for prospective satisficing measures do not provide a complete description of what a PSM may look like. More generally, it is interesting to connect the PSP model to other choice models. We now show how one can represent all PSMs in terms of a more classical definition of a “risk measure.” Risk measures are distinct mathematical entities from PSMs and are motivated and defined in a very different way. Therefore, it seems quite intriguing that they are so closely connected to PSMs.

Following Föllmer and Schied [14], we first formally define the concept of a risk measure.

**Definition 3.** A function \( \mu : \mathcal{X} \to \mathbb{R} \) is a risk measure over \( \mathcal{X} \) if it satisfies the following for all \( X, Y \in \mathcal{X} \):

1. Monotonicity: If \( X \geq Y \), then \( \mu(X) \leq \mu(Y) \).
2. Translation invariance: If \( c \in \mathbb{R} \), then \( \mu(X + c) = \mu(X) - c \).

A risk measure \( \mu \) may be interpreted as the amount of money (“capital”) needed to make a position acceptable by some standard. Namely, \( \mu(X + \mu(X)) = \mu(X) - \mu(X) = 0 \), i.e., adding the capital \( \mu(X) \) to the risky position \( X \), one obtains a new position with “zero risk,” and positions with non-positive risk can be considered as acceptable. In other words, acceptable positions do not require additional, guaranteed capital. We now formalize the concept of acceptable positions.

**Definition 4.** Let \( \mu : \mathcal{X} \to \mathbb{R} \) be a risk measure. The subset \( \mathcal{A}_\mu \) of \( \mathcal{X} \) defined by

\[ \mathcal{A}_\mu = \{ X \in \mathcal{X} : \mu(X) \leq 0 \} \]

is called the acceptance set associated to the risk measure \( \mu \) and \( X \in \mathcal{A}_\mu \) is an acceptable position.

The two properties of risk measures have clear implications for the acceptance set: if one position always pays as much as an acceptable position, then it must be acceptable as well. In addition, if we add a constant payoff to a risky position, then the amount of capital required to make the position acceptable is reduced accordingly. We refer the reader to Föllmer and Schied [15] and the many references therein for more on risk measures and the properties of the corresponding acceptance sets.

Though it is not necessary for the main results, we will assume throughout that the risk measure is normalized; in other words, \( \mu(0) = 0 \). This is without loss of generality due to translation invariance.

The class of convex risk measures has garnered much attention. Formally, we say a risk measure is convex if, for any \( X, Y \in \mathcal{X}, \lambda \in [0,1] \),

\[ \mu(\lambda X + (1 - \lambda)Y) \leq \max\{\mu(X), \mu(Y)\} \] (3)
and \(\text{concave} if\)
\[
\mu(\lambda X + (1 - \lambda)Y) \geq \min\{\mu(X), \mu(Y)\}. \tag{4}
\]

Notice that the preference relation \(\succeq_{\mu}\) induced by a risk measure \(\mu\) will follow \(X \succeq_{\mu} Y\) if and only if \(\mu(X) \leq \mu(Y)\). It is not hard to see that (3) is equivalent to the preference relation \(\succeq_{\mu}\) being diversification favoring (i.e., convex preferences), and (4) is equivalent to \(\succeq_{\mu}\) being concentration favoring. Convex risk measures are usually defined (e.g., Föllmer and Schied [14]) via \(\mu\) satisfying convexity directly, not quasi-convexity as in (3); we now show that this is in fact equivalent.

**Proposition 2.** A risk measure \(\mu\) that is diversification favoring is equivalent to the function \(\mu\) being convex, i.e., for all \(X, Y \in \mathcal{X}\), \(\lambda \in [0, 1]\), \(\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)\). Likewise, concentration favoring is equivalent to the function \(\mu\) being concave.

Typically, convex risk measures are described with convexity (as opposed to quasi-convexity, which, via Proposition 2, implies convexity when the function is translation invariant) directly. We use quasi-convexity in the definition here instead because quasi-convexity of the risk measure leads directly to the notion of convex preferences, and this seems to be the natural way to describe “diversification favoring” (in fact, it is unclear what the convexity property on the function directly implies about the preference relation). We note that convex risk measures are well-studied objects (some examples of convex risk measures are the certainty equivalent under an exponential utility function and conditional value-at-risk, both of which will come up repeatedly in later discussions), whereas concave risk measures are not. We will give some examples of concave risk measures later.

We are now ready to show our main representation result, which states that we may construct all PSMs via families of risk measures. In everything that follows, we will use the convention \(\sup \emptyset = -\infty\).

**Theorem 1.** Consider a function \(\rho : \mathcal{X} \to \mathbb{R} \cup \{-\infty, \infty\}\) in which the neutral, secured and vulnerable sets of target premia are given by (2). Then, \(\rho\) is a prospective satisficing measure if and only if there exists a family of risk measures \(\{\mu_k : k \in (-\infty, \infty)\setminus\{0\}\}\), nondecreasing in \(k\), i.e., \(k \mapsto \mu_k(X)\) is nondecreasing for all \(X \in \mathcal{X}\), convex if \(k \in (0, \infty)\), concave if \(k \in (-\infty, 0)\), and with closed acceptance sets \(\mathcal{A}_{\mu_k}\) for all \(k \in (-\infty, \infty)\setminus\{0\}\), such that
\[
\rho(X) = \sup\{k \in (-\infty, \infty)\setminus\{0\} : \mu_k(X) \leq 0\}. \tag{5}
\]

Moreover, given a prospective satisficing measure \(\rho\), the underlying risk measure for \(k \in \mathbb{R}\setminus\{0\}\) is given by
\[
\mu_k(X) = \inf\{a : \rho(X + a) \geq k\}. \tag{6}
\]

Note that Equation (5) is equivalent to
\[
\rho(X) = \sup\{k \in (-\infty, \infty)\setminus\{0\} : X \in \mathcal{A}_{\mu_k}\}.
\]

Moreover, inspection of the proof of Theorem 1 shows that
\[
\mathcal{A}_{\mu_k} = \{X \in \mathcal{X} : \rho(X) \geq k\}.
\]
In words, $X \in \mathcal{X}$ is acceptable with respect to $\mu_k$ if and only if $\rho(X) \geq k$.

An intuition for the representation theorem can be gained as follows. If risk measures are used for selecting among positions, decision makers first need to choose an index parameter (i.e., the $k$ in the notation above) and then find $X \in \mathcal{X}$ with the smallest risk, $\mu_k(X)$ (subject to some other constraints, perhaps). The index $k$ can be seen as an aversion level since the family of risk measures $\{\mu_k : k \in (-\infty, \infty) \setminus \{0\}\}$ is non-decreasing in $k$, i.e., if $X$ has non-positive risk under $k$ (and thus $X \in \mathcal{A}_{\mu_k}$, so $X$ is acceptable under $\mu_k$) then it also has non-positive risk under all $k' \leq k$ (and thus $X \in \mathcal{A}_{\mu_{k'}}$, so $X$ is also acceptable under $\mu_{k'}$). In other words, decision makers with a smaller index are willing to accept all positions that are accepted by decision makers with larger $k$ and thus are more risk tolerant. As a concrete example of this, $\mu_k$ could be the negative of the certainty equivalent under an exponential utility function (CARA), with $k$ being the reciprocal of the risk tolerance parameter. We refer to this as the entropic risk measure and will discuss it more in the next section.

When using a PSM, however, decision makers specify a target payoff but do not need to pick an index parameter. They will instead rank positions according to the maximum possible index such that the risk of falling short of the target at that index value is still acceptable. If a position has a positive PSM value, then the position is “secured” relative to the target; its risk is acceptable to any risk averse investor with an index up to $\rho(X) > 0$. On the other hand, if the position has negative PSM, the position is “vulnerable” relative to the target and is only acceptable to investors who are sufficiently risk seeking, i.e., those with an index no bigger than $\rho(X) < 0$. We will provide more intuition on PSMs when we discuss some specific examples in the next section.

2.2 Convex and concave risk measures and associated PSMs

As noted, convex risk measures are well-known objects; this is not the case for concave risk measures. We can, however, easily construct concave risk measures from convex ones. We now discuss this.

**Proposition 3.** Consider a family of convex risk measures $\{\mu_k : k \in (0, \infty)\}$ that is non-decreasing on $k \in (0, \infty)$. Let $\tilde{\mu}_k(X) = -\mu_{-k}(-X)$ for all $X \in \mathcal{X}$ and $k \in (-\infty, 0)$. Then, the family of risk measures $\{\tilde{\mu}_k : k \in (-\infty, 0)\}$ is concave and non-decreasing on $k \in (-\infty, 0)$. Moreover,

$$\tilde{\mu}_s(X) \leq \mu_t(X) \quad \forall s < 0, t > 0.$$  

For our examples of PSMs, we will primarily focus on families of risk measures where the concave risk measures ($k < 0$) are derived from convex ones ($k > 0$) using the construction of Proposition 3. This motivates the following definition.

**Definition 5.** We say that a family of risk measures $\{\mu_k : k \in (-\infty, \infty) \setminus \{0\}\}$ has symmetric properties if

$$\mu_k(X) = -\mu_{-k}(-X)$$

for all $X \in \mathcal{X}$ and $k \in \mathbb{R} \setminus \{0\}$.

PSMs generated by families of risk measures with symmetric properties also possess symmetric properties under additional, mild conditions, as shown in the following proposition.
Proposition 4. Let \( \{ \mu_k : k \in (-\infty, \infty) \setminus \{0\} \} \) be a family of risk measures with symmetric properties, such that for all non-deterministic \( X \in \mathcal{X} \) the function \( k \mapsto \mu_k(X) \) is strictly increasing.\(^4\) Let \( \rho \) be the corresponding PSM. Then for all \( X \neq 0 \) we have
\[
\rho(-X) = -\rho(X).
\]

We now provide some examples of concave risk measures derived from convex counterparts, as well as the associated PSMs. In the first three examples, we will assume knowledge of an underlying probability measure \( \mathbb{P} \) (this could be objective or subjective). In general, it is not necessary to have a pre-specified probability measure, as can be seen from above. We will touch upon this issue further when we discuss PSMs in the context of ambiguity.

Example 1 (Entropic PSM (EPSM)). The family
\[
\mu_k(X) = \frac{1}{k} \ln \mathbb{E}[\exp(-kX)] \quad k \neq 0,
\]
is a symmetric family of nondecreasing risk measures that are convex for \( k > 0 \). The associated PSM is given by
\[
\rho(X) = \sup \left\{ k \in (-\infty, \infty) \setminus \{0\} : \frac{1}{k} \ln \mathbb{E}[\exp(-kX)] \leq 0 \right\},
\]
which we call the entropic prospective satisficing measure (EPSM). If \( X \) is normally distributed with mean \( \mu \) and standard deviation \( \sigma \) under the probability \( \mathbb{P} \), then we have \( \mu_k(X) = -\mu + k\sigma^2/2 \), which rewards (i.e., has less “risk”) for greater variance. In this case, we have \( \rho(X) = 2\mu/\sigma^2 \). Note that the secured set is those positions with positive mean, and the vulnerable set is those with negative means. For a fixed, positive mean, we prefer smaller variance (risk aversion). If the mean is negative, however, we prefer larger variance; in this case, the intuition remains that larger variance gives us better hopes of attaining the target.

Note that the positive part of the above representation yields the entropic satisficing measure of Brown and Sim \[5\], which is also the reciprocal of the riskiness index of Aumann and Serrano \[3\].

Example 2 (Conditional value-at-risk (CVaR) PSM). The family
\[
\mu_k(X) = \begin{cases} 
\text{CVaR}_{\epsilon-k}(X) & \text{if } k > 0 \\
-\text{CVaR}_{\epsilon+k}(-X) & \text{if } k < 0 
\end{cases}
\]
where
\[
\text{CVaR}_\epsilon(X) = \inf_{\nu \in \mathbb{R}} \left\{ \nu + \frac{1}{\epsilon} \mathbb{E} \left[ (-X - \nu)^+ \right] \right\}
\]
is a symmetric family of nondecreasing risk measures that are coherent\(^5\) for \( k > 0 \). The PSM given by this symmetric family is
\[
\rho(X) = \begin{cases} 
\sup \left\{ k > 0 : \text{CVaR}_{\epsilon-k}(X) \leq 0 \right\} & \text{if } \mathbb{E}[X] \geq 0, \\
\sup \left\{ k < 0 : \text{CVaR}_{\epsilon+k}(-X) \geq 0 \right\} & \text{otherwise},
\end{cases}
\]

\(^4\)We say that \( X \in \mathcal{X} \) is non-deterministic, if there exist events \( A, B \in \mathcal{F} \) such that \( X(\omega) \neq X(\omega') \) for \( \omega \in A, \omega' \in B \).

\(^5\)In addition to monotonicity, translation invariance and convexity, a coherent risk measure \( \mu_k \) also satisfies the positive homogeneity property, which states that \( \mu(\lambda X) = \lambda \mu(X) \) for all \( X \in \mathcal{X} \) and \( \lambda \geq 0 \).
which we call the CVaR PSM. A variant of CVaR measure (without the risk seeking part and scaled to be on \((0, 1]\)) is defined in Brown and Sim [5]. When \(X\) is normally distributed under \(P\), we have
\[
\rho(X) = \begin{cases} 
\sup \left\{ k > 0 : \frac{\phi(\Phi^{-1}(e^{-k}))}{e^{-k}} \sigma(X) \leq \mathbb{E}[X] \right\} & \text{if } \mathbb{E}[X] \geq 0, \\
\sup \left\{ k < 0 : \frac{\phi(\Phi^{-1}(e^k))}{e^k} \sigma(X) \leq -\mathbb{E}[X] \right\} & \text{otherwise},
\end{cases}
\]
where \(\phi\) and \(\Phi\) are the standard normal density and cumulative distribution functions, respectively.

The PSM in this case is a monotonic transformation of the Sharpe ratio (i.e., \(\mathbb{E}[X]/\sigma(X)\)).

Example 3 (Homogenized entropic PSM (HEPSM)). The family
\[
\mu_k(X) = \begin{cases} 
\inf_{a > 0} \{ a \ln (\mathbb{E}[\exp(-X/a)]) + ak \} & \text{if } k > 0 \\
\sup_{a < 0} \{ a \ln (\mathbb{E}[\exp(-X/a)]) - ak \} & \text{if } k < 0
\end{cases}
\]
is a symmetric family of nondecreasing risk measures that are coherent for \(k > 0\). The associated PSM is given by
\[
\rho(X) = \begin{cases} 
\sup \left\{ k > 0 : \inf_{a > 0} \{ a \ln \mathbb{E}[\exp(-X/a)] + ak \} \leq 0 \right\} & \text{if } \mathbb{E}[X] \geq 0, \\
\sup \left\{ k < 0 : \sup_{a < 0} \{ a \ln \mathbb{E}[\exp(-X/a)] - ak \} \leq 0 \right\} & \text{otherwise},
\end{cases}
\]
If \(X\) is normally distributed under \(P\), then
\[
\rho(X) = \frac{\text{sign}(\mathbb{E}[X])}{2} \left( \frac{\mathbb{E}[X]}{\sigma(X)} \right)^2,
\]
which is again a monotonic transformation of the Sharpe ratio.

Example 4 (General symmetric family). Let \(\alpha_k\) be a family of convex functions on the space \(Q\) of probability measures on \((\Omega, \mathcal{F})\), such that \(\alpha_k\) is non-increasing in \(k\), and satisfying \(\alpha_{-k} = -\alpha_k\) for all \(k \in \mathbb{R}\). It is well-known (e.g., Föllmer and Schied [15]) that the family of risk measures defined by
\[
\mu_k(X) = \sup_{Q \in Q} \{-\mathbb{E}_Q[X] - \alpha_k(Q)\}, \quad (7)
\]
for \(k > 0\) is a family of convex risk measures. This family in turn generates the concave family
\[
\bar{\mu}_k(X) = \sup_{Q \in Q} \{-\mathbb{E}_Q[X] - \alpha_{-k}(Q)\} = -\sup_{Q \in Q} \{\mathbb{E}_Q[X] + \alpha_k(Q)\} = \inf_{Q \in Q} \{-\mathbb{E}_Q[X] - \alpha_k(Q)\},
\]
which can be interpreted as the (negative of the) best case penalized expected value over all distributions. The corresponding PSM can be expressed as
\[
\rho(X) = \max \left\{ \sup_{Q \in Q} \left\{ k > 0 : \inf_{Q \in Q} \{\mathbb{E}_Q[X] + \alpha_k(Q)\} \geq 0 \right\} , \sup_{Q \in Q} \left\{ k < 0 : \sup_{Q \in Q} \{\mathbb{E}_Q[X] + \alpha_k(Q)\} \geq 0 \right\} \right\}.
\]

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2.3 A distributional perturbation view of prospective satisficing

Example 4 suggests a general interpretation of any prospective satisficing measure. As discussed, (7) defines a convex risk measure. In fact, any convex risk measure $\mu_k$ can be expressed as

$$\mu_k(X) = \sup_{Q \in \mathcal{Q}} \{-E_Q[X] - \alpha_k(Q)\},$$

where $\alpha_k : \mathcal{Q} \rightarrow \mathbb{R}$ is a nonnegative convex function on the set $\mathcal{Q}$ of probability measures on $(\Omega, \mathcal{F})$. For simplicity, let us focus on symmetric PSMs. From the example, if $\rho(X) > 0$, we can express it as

$$\rho(X) = \sup \left\{ k > 0 : \inf_{Q \in \mathcal{Q}} \{E_Q[X] + \alpha_k(Q)\} \geq 0 \right\},$$

where $\alpha_k$ is non-increasing in $k$. This has a natural robustness interpretation in terms of hitting the target in expectation for all distributions on the probability space. Specifically, one can imagine that the distribution of $X$ is being chosen by an adversary who is trying to reduce the expected value of $X$; this adversary has to pay a nonnegative penalty $\alpha_k(Q)$, however, for any distribution $Q$ they choose. As $k$ increases, $\alpha_k$ gets smaller and therefore the adversary has more power via a reduced penalty. $\rho(X) > 0$ then represents the most power we can give this adversary such that at that level we still hit the target in (penalized) expectation for all possible distributions. In this case, $\rho(X)$ represents an index of robustness or security.

On the other hand, if $\rho(X) < 0$, we can write the PSM as

$$\rho(X) = \sup \left\{ k < 0 : \sup_{Q \in \mathcal{Q}} \{E_Q[X] + \bar{\alpha}_k(Q)\} \geq 0 \right\},$$

where $\bar{\alpha}_k = -\alpha_{-k}$ is a non-positive, concave function. The interpretation here is quite different: rather than being chosen by an adversary, the distribution is being selected by an ally who is looking to find any distribution such that $X$ beats the target in expectation. In this case, the expectation is reduced by the penalty $\bar{\alpha}_k(Q)$, which is nondecreasing on $k < 0$. Now, $\rho(X) < 0$ denotes the largest index level at which we can penalize this ally such that they can still find distribution under that $X$ hits the target in (penalized) expectation. Here $X$ is in the vulnerable set, so the investor cannot hope to be robust; they are simply looking for some hope of beating the target. In this case, we can think of $\rho(X)$ as signifying an index of fragility or vulnerability.

As a concrete example of this, consider the entropic prospective satisficing measure (Example 1) for a random variable with distribution $\mathbb{P}$. It is well-known that the entropic risk measure

$$\mu_k(X) = \frac{1}{k} \ln E \left[ e^{-kX} \right]$$

is generated by the convex penalty function

$$\alpha_k(Q) = \frac{1}{k} E_Q \left[ \ln \left( \frac{dQ}{d\mathbb{P}} \right) \right],$$

i.e., $k^{-1}$ times the relative entropy from $Q$ to $\mathbb{P}$ for all distributions $Q$ absolutely continuous with respect to $\mathbb{P}$ (denoted $Q \ll \mathbb{P}$). Here, $\rho(X) > 0$ means the lower bound

$$E_Q[X] \geq -\frac{1}{k} E_Q \left[ \ln \left( \frac{dQ}{d\mathbb{P}} \right) \right]$$
holds for all distributions $Q \ll P$ on $\Omega$ for any $k \in (0, \rho(X)]$. $\rho(X) < 0$, on the other hand, means the above lower bound holds for some distribution $Q \ll P$ for any $k \in (-\infty, \rho(X)]$.

This interpretation in terms of expected (penalized) payoffs under ambiguity seems interesting in light of Simon’s [37] contention that probabilistic information is often limited for real-world decision makers.

### 2.4 Stochastic dominance properties

The PSP properties related to satisficing (i.e., attainment content and non-attainment apathy) focus only on whether a position beats the target or not, but not by how much. As a result, one may think that in cases when we do have an underlying probability measure $P$ with respect to which stochastic orders can be defined, it is possible that PSMs violate first-order stochastic dominance (FSD). In this section, we show that provided the underlying risk families generating the PSM obey FSD, then so will the PSM itself. Moreover, for risk measures satisfying second-order stochastic dominance (SSD), the corresponding PSM will obey SSD over the secured sets and risk-seeking stochastic dominance (RSSD) over the vulnerable sets.

Finally, we show that these properties will be obeyed by most PSMs of practical interest: namely, those that depend only on the distribution of the random variable under $P$. In these cases, we also have an interpretation for the secured and vulnerable positions as simply those with positive and negative means, respectively.

We first recall the definition of the stochastic orders just mentioned. Note that in this section, if not specified explicitly, expectations are taken with respect to the probability measure $P$. We say that $X$ dominates $Y$ by FSD if and only if $E[u(X)] \geq E[u(Y)]$ for all nondecreasing functions $u$ and the inequality is strict for at least one such $u$; in this case we write $X \geq_{(1)} Y$. Similarly, $X$ dominates $Y$ by SSD (respectively RSSD) if and only if $E[u(X)] \geq E[u(Y)]$ for all $u$ nondecreasing and concave (respectively convex) and the inequality is strict for at least one such $u$; in this case we write $X \geq_{(2)} Y$ (respectively $X \geq_{(-2)} Y$). Alternative equivalent definitions of first order, second order and risk-seeking stochastic dominance can be found in Levy [25].

We first provide a result that will be very helpful for establishing stochastic dominance in the case of PSPs arising from families of risk measures with symmetric properties.

#### Proposition 5

Let $\mu$ be a risk measure and suppose that $\mu$ preserves FSD, i.e., if $X \geq_{(1)} Y$ then $\mu(X) \leq \mu(Y)$. Then the risk measure $\bar{\mu}(X) = -\mu(-X)$ also preserves FSD. Moreover, if $\mu$ preserves SSD, then $\bar{\mu}$ preserves RSSD.

The proposition shows that for symmetric families of risk measures, preservation of first order stochastic dominance for the convex risk measures $\mu_k$ implies preservation of first order stochastic dominance for concave risk measures $\bar{\mu}_k$. In addition, when second order stochastic dominance holds for $\mu_k$, then risk-seeking stochastic dominance follows for $\bar{\mu}_k$. In other words, in the case of symmetric families, we only need to specify stochastic dominance properties for the convex risk measure $\mu_k$ in order to characterize stochastic dominance properties for the whole family of risk measures $\{\mu_k : k \in [-\infty, \infty) \setminus \{0\}\}$. Given that convex risk measures are well-studied, Proposition 5 will allow us to use well-known results to characterize stochastic dominance properties of PSMs.
For this step, we first need to show that stochastic dominance properties for PSMs are implied by those of the associated family of risk measures. We now show this.

**Proposition 6.** Let \( \{\mu_k : k \in (-\infty, \infty) \setminus \{0\}\} \) be a nondecreasing family of risk measures and let \( \rho \) be the associated PSM. Suppose that \( \mu_k \) preserves FSD for all \( k \). Then \( \rho \) preserves FSD, i.e.,

\[
X \geq_{(1)} Y \Rightarrow \rho(X) \geq \rho(Y).
\]

Moreover, if \( \mu_k \) preserves SSD for \( k > 0 \) and RSSD for \( k < 0 \), then

\[
\forall X \in \mathcal{X}, Y \in \mathcal{X}^{++} \text{ such that } X \geq_{(2)} Y \Rightarrow \rho(X) \geq \rho(Y) \quad \forall X \in \mathcal{X}, Y \in \mathcal{X}^{-} \text{ such that } X \geq_{(-2)} Y \Rightarrow \rho(X) \geq \rho(Y).
\]

In general, convex risk measures do not preserve FSD or SSD, as shown by De Giorgi [9] for the case of coherent risk measures. Therefore, Proposition 6 is of little help if we do not specify the conditions on the family of risk measures \( \mu_k \) such that stochastic dominance is preserved. It is well known that stochastic dominance orders are fully characterized by the cumulative distribution functions of the corresponding random variables under the specified underlying probability measure \( P \) (see Levy [25]).

When a risk measure \( \mu \) does not only depend on the distribution functions of the prospects, we can find two prospects \( X \) and \( Y \) that only differ on zero-probability events, but possess different values for the risk measure, e.g., \( \mu(X) > \mu(Y) \). In this case, we can define a third prospect \( Z = X + \epsilon, \) \( 0 < \epsilon < \mu(X) - \mu(Y) \), which obviously dominates \( X \) by FSD (and thus also dominates \( Y \) by FSD), but \( \mu(X) > \mu(X) - \epsilon = \mu(Z) \) and \( \mu(Z) = \mu(X) - \epsilon > \mu(Y) \). This shows that a necessary property on risk measures in order to have preservation of stochastic dominance orders is that they only depend on the probability distribution of the prospect. We thus introduce the following definition:

**Definition 6.** Let \( P \) be a probability measure on \( (\Omega, \mathcal{F}) \). A function \( f : \mathcal{X} \rightarrow \mathbb{R} \) is called law-invariant (with respect to \( P \)) if and only if \( f(X) = f(Y) \) whenever \( X \) and \( Y \) have the same distribution under \( P \), i.e., \( P\{X(\omega) \leq x\} = P\{Y(\omega) \leq x\} \) for all \( x \in \mathbb{R} \).

Law-invariance means the underlying mapping between the event space and the premia space is irrelevant; all that matters is the distribution of the premia under \( P \). It also means that zero-probability events do not matter, i.e., it might be that two random variables differ on events \( A \subset \Omega \), but as long as \( P\{A\} = 0 \), this does not have any impact on the function \( f \). This is the case for every PSM we discuss in the context of random variables with given distributions and seems like an eminently reasonable property, common to most models of decision making under uncertainty.

In our context, law-invariance is very useful because it has implications for stochastic dominance.

**Proposition 7.** Let \( (\Omega, \mathcal{F}, P) \) be a atomless probability space.\(^6\) If \( \rho \) is a law-invariant PSM, then \( \rho \) preserves FSD on \( \mathcal{X} \), preserves SSD on \( \mathcal{X}^{++} \) (secured positions) and RSSD on \( \mathcal{X}^{-} \) (vulnerable positions).

\(^6\)A probability space \( (\Omega, \mathcal{F}, P) \) is said to be atomless if there exist no \( \omega \in \Omega \) such that \( P\{\omega\} > 0 \).
On atomless probability spaces, law-invariant risk measures also display important boundedness properties. We say that a convex (concave) risk measure $\mu$ is bounded from below (above) by the expectation when $\mu(X) \geq \mathbb{E}[-X]$ ($\mu(X) \leq \mathbb{E}[-X]$) for all $X \in \mathcal{X}$. On atomless probability spaces, Föllmer and Schied [15] show that law-invariant convex risk measures are bounded from below by the expectation. This implies that concave risk measures are bounded from above by the expectation. Namely, if $\mu$ is law-invariant and concave then $\mu(X) = -\mu(-X)$ is law-invariant and convex, thus $\mu(X) \geq \mathbb{E}[-X]$, or equivalently, $\mu(X) \leq \mathbb{E}[-X]$. We now show that bounded properties have important implications for the structure of the secured and vulnerable sets:

**Theorem 2.** Let $\{\mu_k : k \in (-\infty, \infty) \setminus \{0\}\}$ be a nondecreasing family of risk measures and let $\rho$ be the associated PSM. Assume that for $k > 0$, $\mu_k$ is bounded from below by the expectation and for $k < 0$, $\mu_k$ is bounded from above by the expectation. Then

$$
\mathbb{E}[X] < 0 \Rightarrow \rho(X) \leq 0
$$

$$
\mathbb{E}[X] \geq 0 \Rightarrow \rho(X) \geq 0.
$$

If the probability space is not atomless, then it is generally not true that a convex (concave) risk measure is bounded from below (above). This property has been added by Rockafellar et al. [35] as an additional property on convex risk measures in order to define their class of “deviation measures.” In many cases, however, convex (concave) risk measures are bounded from below (above) even if the probability space is not atomless. This is the case for the convex risk measures of Examples 1-3 discussed in Section 2.2, as shown in the following proposition.

**Proposition 8.** The underlying families of risk measures in EPSM, CVaR PSM and HEPSM are bounded by the expectation, i.e., $\mu_k(X) \geq \mathbb{E}[-X]$ for $k > 0$ and $\mu_k(X) \leq \mathbb{E}[-X]$ for $k < 0$.

We point out the generality of Theorem 2, which applies not only to any law-invariant PSM on an atomless probability space, but also to several PSMs on non-atomless spaces as seen in Proposition 8. Theorem 2 implies that secured target premia possess a nonnegative expected value, i.e., on average they are at least as good as the target. In contrast, vulnerable target premia possess a negative expected value, i.e., on average they fail to outperform the target. This provides a simple interpretation of the “secured” and “vulnerable” sets, respectively.

Theorem 2 also shows an interesting practical advantage of PSMs relative to models of decision making under uncertainty involving utility functions or subjective distortions of the probability space. In particular, consider $X, Y \in \mathcal{X}$ with $\mathbb{E}[X] > 0 > \mathbb{E}[Y]$. Note that in order to compute these expected values we only need to specify the target $\tau$ and nothing else about the structure of the underlying risk measures. Theorem 2 implies that any decision maker using a law-invariant PSM, where the corresponding family of risk measures satisfies the boundedness properties, will either prefer $X$ to $Y$ or be indifferent between the two. Thus, $X$ can be taken as the (weakly) preferred prospect in all cases. For expected utility maximizers, by contrast, all rankings are possible: the ranking will depend on the specific structure of the decision maker’s utility function, which must therefore first be specified.

In such settings, therefore, the decision maker using a PSM can immediately simplify their decision problem by disregarding prospects with negative expected payoffs.

18
3 PSPs and connections to Allais

In this section, we show how choice under PSPs can be consistent with some of the classical patterns of observed preferences noted famously by Allais [1]. Such observed preferences are impossible under classical expected utility theory due to a strong requirement known as the independence axiom. In contrast, PSPs can consistently “explain” such choices over fairly wide ranges of targets.

We begin with a specific pair of gambles and show computational results using several PSMs, then shift our focus to more general statements using the entropic PSM.

3.1 A specific Allais example

Consider the following two sets of gambles:

- *Gamble A:* Wins $500,000 for sure.
- *Gamble B:* 1% chance of 0, 10% chance of winning $2,500,000 and 89% chance of winning $500,000.

and

- *Gamble C:* 90% chance of 0, 10% chance of winning $2,500,000.
- *Gamble D:* 89% chance of 0, 11% chance of winning $500,000.

The most typical pattern of preferences observed among actual decision makers is to choose A over B and C over D. It is not hard to see that this is inconsistent with traditional expected utility theory with any utility function.

In contrast, these choice pairs can in fact be consistent with choice under a PSM over a specific range of a fixed target. For instance, let $\rho$ be any law-invariant PSM and let $\tau$ be the target. We further assume that the corresponding family of risk measures satisfies the boundedness properties of Theorem 2. Denoting gamble A by $X_A$, we have, for $\tau \leq 500,000$, $\rho(X_A - \tau) = \infty$, so $X_A - \tau \succeq X_B - \tau$. On the other hand, the expected value of gamble C is $250,000$ and the expected value of gamble D is $55,000$; therefore, using Theorem 2, for $\tau > 55,000$, $\tau < 250,000$, we have $\rho(X_C - \tau) \geq 0 \geq \rho(X_D - \tau)$, so the observed pattern above is (weakly) resolved over $\tau \in (75,000, 250,000)$.

In fact, for many PSMs, this pattern of preferences will be observed over an even larger range of targets. This is shown in Tables 1-3 for EPSM, HEPSM, and CVaR PSM, respectively.

In all three cases, gamble A is strictly preferred to gamble B for $\tau \leq 500,000$, and gamble C is strictly preferred to gamble D for $\tau \in (\tau_z, 2,000,000)$, for some $0 < \tau_z < 55,000$. The intuition in the first pair is that gamble A is guaranteed to hit the $500,000 target; for the second pair, as long as the target is not very small, the extra “upside” of $2,500,000 versus $500,000 outweighs the small difference in probabilities of zero payoffs. It seems plausible that this type of intuition is being used by the decision makers who make such choices.\(^7\)

\(^7\)It is worth mentioning that the imposition of some risk-seeking behavior, as we have done, is unnecessary to address the example of Allais; the resolution would still hold without this, but over a target range that is quite smaller.
3.2 Common consequence effect

We now briefly generalize the above pattern over a pair of choices. The effect above, first pointed out by Allais [1], is typically called the common consequence effect.

Formally, consider two positive payoffs \( x > y > 0 \) and two probabilities \( q \in (0, 1), p \in (0, 1) \), with \( q > p \). As before, we denote the first pair of gambles \( X_A \) and \( X_B \). \( X_A \) is a sure payoff of \( y \); \( X_B \), on the other hand, pays \( x \) with probability \( p \), \( y \) with probability \( 1 - q \), and 0 with probability \( q - p \).

The second pair is a pair of all-or-nothing gambles, which we denote \( X_C \) and \( X_D \). \( X_C \) pays \( x \) with probability \( p \) and 0 otherwise; \( X_D \) pays \( y \) with probability \( q \) and 0 otherwise. We will assume \( X_C \) beats \( X_D \) in expectation, i.e., \( px > qy \), though we could remove this assumption in what follows.

In observed choices, particularly when \( x \) is considerably larger than \( y \) and \( q - p \) is small, real-world decision makers often prefer the “safer” choice among the first two gambles (i.e., the sure payoff of \( X_A \) over the risky payoff \( X_B \)) and the “riskier” choice among the second two gambles (i.e., \( X_C \) over \( X_D \)).

The rationale, presumably, is along the lines that \( X_A \) offers a sure payoff, whereas \( X_B \) can result in a zero payoff; for the second pair, though \( X_C \) has a slightly higher chance of paying off nothing, this extra risk may well be worth bearing if the difference \( x - y \) is large.

This is easily seen to be inconsistent with expected utility theory. Let \( u \) be any utility function, normalized to \( u(0) = 0 \). Then strict preference of \( X_A \) over \( X_B \) implies \( u(y) > pu(x) + (1 - q)u(y) \Rightarrow qu(y) > pu(x) \); on the other hand, strict preference of \( X_C \) over \( X_D \) implies \( pu(x) > qu(y) \), a contradiction. This occurs, fundamentally, because of the independence axiom, which imposes the requirement that common components of any two gambles be irrelevant to the direction of preference. In effect, this strong requirement forces decision makers to be unaffected by the context surrounding their choices.

Let us make this more specific. If we consider a gamble \( X_E \) that pays \( x \) with probability \( p/q \) and 0 with probability \( (q - p)/q \). Then we see that \( X_B \) is the gamble that pays \( X_A \) with probability \( 1 - q \) and \( X_E \) with probability \( q \). Similarly, \( X_C \) pays \( X_E \) with probability \( q \) and 0 with probability \( 1 - q \), and \( X_D \) pays \( X_A \) with probability \( q \) and 0 with probability \( 1 - q \). With this decomposition, we see that the independence axiom requires that \( X_A \) be (strictly) preferred to \( X_B \) if and only if \( X_A \) be (strictly) preferred to \( X_E \), and \( X_C \) be (strictly) preferred to \( X_D \) if and only if \( X_E \) be (strictly) preferred to \( X_A \). Thus, strict preference of \( X_A \) over \( X_B \) and \( X_C \) over \( X_D \) violates the independence axiom.

We show that the common consequence effect can be explained by PSM over an explicit range of targets; we show a formal result for entropic PSM. As in the example in the previous subsection, the result will hold for other PSMs, too.

**Proposition 9.** Consider the two pairs of gambles, \((X_A, X_B)\) and \((X_C, X_D)\), as described above with \( px > qy \) and let \( \rho \) denote the entropic prospective satisficing measure and \( \mu_k \) denote the entropic risk measure at level \( k \). Then for every \((x, y, p, q)\) as above, there exists a target \( \tau(x, y, p, q) = \tau^* \) \( < qy \) such that for all \( \tau \in (\tau^*, y] \), \( \rho(X_A - \tau) > \rho(X_B - \tau) \) and \( \rho(X_C - \tau) > \rho(X_D - \tau) \). Moreover, we have

\[
\tau^* = -\mu_{\rho^*}(X_D),
\]

where \( \rho^* \) is the unique \( \rho > 0 \) such that \( \mu_{\rho}(X_C) = \mu_{\rho}(X_D) \).

Notice that the entropic PSM is closely linked to an expected utility representation with an exponential function (CARA utility). Despite this close connection, however, the implications for choice may be drastically different than those implied by expected utility theory.
3.3 Common ratio effect

The common ratio effect is another well-known pattern of many observed preferences, again famously pointed out by Allais [1], that cannot be captured by expected utility theory. This phenomenon is again found in the preferences typically observed over two pairs of gambles. Consider two positive real numbers \( x > y > 0 \) and two probabilities \( q \in (0, 1), p \in (0, 1) \).

We denote the first pair of gambles by \( X_A \) and \( X_B \). The first, \( X_A \), involves a sure bet of \( y \). The second, \( X_B \), pays off \( x \) with probability \( p \) and \( 0 \) with probability \( 1 - p \).

The second pair of gambles, \( X_C \) and \( X_D \), involves two risky bets: \( X_C \) pays off \( x \) with probability \( p(1 - q) \) and \( 0 \) otherwise; \( X_D \) pays off \( y < x \) with probability \( 1 - q \) and \( 0 \) otherwise. Note that we can view both of these as a composition of two biased coin flips; first, a coin with probability \( 1 - q \) of getting a head (i.e., a positive payoff), then a coin with probability \( p \) of getting a head. \( X_C \) pays off \( x \) if both coins get heads; \( X_D \) pays off \( y \) if the first coin gets a head regardless of the outcome of the second coin. Note that, conditional on the first coin getting a head, \( X_C \) and \( X_D \) offer the exact same gambles as \( X_A \) and \( X_B \).

It is not uncommon for real-world decision makers to prefer \( X_A \) over \( X_B \) while also preferring \( X_C \) over \( X_D \). For instance, consider \( p = .8, q = .75, x = 16, \) and \( y = 10 \). Many subjects prefer the sure payoff of \( y = 10 \) over the 80\% chance of \( x = 16 \) in the first case; in the second case, however, even though \( X_C \) offers a lower chance of a positive payoff (20\% vs. 25\%), the extra upside of \( x = 16 \) vs. \( y = 10 \) is well worth the extra 5\% chance of zero payoff.

Again, it is easy to see that such a pattern of preferences is inconsistent with expected utility theory. Normalizing \( u(0) = 0 \), the first preference implies \( u(y) > p u(x) \); the second preference implies \( p(1 - q) u(x) > (1 - q) u(y) \Rightarrow p u(x) > u(y) \), a contradiction. Once again, expected utility theory (via the independence axiom) imposes the strong requirement that “common components” of gambles must be irrelevant across sets of choices, and that to make choices otherwise is irrational. In this case, it hardly seems irrational to take into account common components. If there is already a high chance of getting zero payoff, as is true in the second pair of gambles, it does not necessarily seem unreasonable for a decision maker to be willing to tolerate a slightly higher risk of no payoff for substantially enough higher upside. This may not, however, be true in the first case, when \( X_A \) is sure to payoff \( y \).

We now show that this effect can in general be captured by PSM over an explicit range of targets. As with the common consequence effect, we prove the result for entropic PSM.

**Proposition 10.** Consider the two pairs of gambles, \( (X_A, X_B) \) and \( (X_C, X_D) \), as described above and let \( \rho \) denote the entropic prospective satisficing measure and \( \mu_k \) denote the entropic risk measure at level \( k \). Then for every \( (x, y, p, q) \) as above, there exists a target \( \tau^*(x, y, p, q) = \tau^* < y \) such that for all \( \tau \in (\tau^*, y) \), \( \rho(X_A - \tau) > \rho(X_B - \tau) \) and \( \rho(X_C - \tau) > \rho(X_D - \tau) \). Moreover, we have

\[
\tau^* = -\mu_{\rho^*}(X_D) \\
\rho^* = \rho(X_B - y).
\]

Notice that when \( px = y \), i.e., both pairs of gambles are equal in expectation, we obtain \( \rho^* = 0 \) and thus \( \tau^* = E[X_D] = (1 - q)y \). When \( px > y \), the decision maker will prefer \( X_C \) over \( X_D \) for some targets strictly less than \( (1 - q)y \). When \( px < y \), there is still a range of targets for which \( X_C \) is preferred,
Figure 1: The threshold targets $\tau^*$ from Proposition 10 for various $(x, p)$ for the case $y = 10$, $q = .75$. The solid black line shows the crossover value of $E [X_D] = (1 - q) y = 2.5$ for which the satisficing prospective measure switches from risk avoidance below to risk seeking above.

but $\tau^* > (1 - q) y$; in this case, the risk-seeking part of the satisficing measure is at work, and decision makers must have an appreciably high target such that the extra upside provided by $X_C$ is worth it.

Figure 1 shows a numerical example of how $\tau^*$ varies as $x$ and $p$ vary for the case $y = 10$ and $q = .75$. As $x$ or $p$ get larger, the range of targets over which $X_C$ is preferred to $X_D$ becomes wider, which squares with intuition.

3.4 Reflection effect

The common ratio effect as discussed above asserts that increasing the chances of the worst outcome (i.e., zero above) tends to push people from the “safer” choice to the “riskier” choice. Kahneman and Tversky [21], in their seminal paper, have observed in such situations that if all payoffs are reflected around zero, that the opposite pattern of preference reversal occurs: namely, an uncertain chance of a larger loss tends to be preferred to a sure loss of smaller magnitude, but as the chances of a zero (no loss) outcome are increased in equal proportion, subjects tend to reverse preferences towards the “less risky” choice. Obviously, this preference reversal is equally inconsistent with expected utility theory.

The numbers used by Kahneman and Tversky [21] in their Problems 3 and 4 to identify the reflection effect are $y = 3,000$, $x = 4,000$, $p = .8$, and $q = .75$. The following preferences were typically observed:

\[
X_A = (3000, 1) \succ X_B = (4,000, .8)
\]
\[
X_C = (3000, .25) \prec X_D = (4,000, .2),
\]

where $(x, p)$ is a prospect paying $x$ with probability $p$ and 0 else, contrasted with the following, which were also typically observed:

\[
-X_A = (-3,000, 1) \prec -X_B = (-4,000, .8)
\]
\[
-X_C = (-3,000, .25) \succ -X_D = (-4,000, .2).
\]

The gambles $-X_A, -X_B, -X_C$ and $-X_D$ are the reflections around zero of gambles $X_A, X_B, X_C$ and $X_D$, respectively. We now argue that this alternation of preferences is possible using symmetric PSMs
with negative targets. In particular, recall that the family of risk measures $\mu_k$ has symmetric properties if $\mu_{-k}(-X) = -\mu_k(X)$ for all $k \in \mathbb{R} \setminus \{0\}$. If the PSM is given by a family of risk measures with symmetric properties, then Proposition 4 shows that $\rho(-X) = -\rho(X)$ for $X \neq 0$ if the function $k \rightarrow \mu_k(X)$ is strictly increasing for non-deterministic $X$. In this case, then, if the following (as, for example, in Proposition 10 above) holds,

$$\forall \tau \in (\tau^*, y) : \left[\begin{array}{c} \rho(X_A - \tau) > \rho(X_B - \tau) \\
\rho(X_C - \tau) > \rho(X_D - \tau) \end{array}\right],$$

then

$$\forall \tau \in (-y, -\tau^*) : \left[\begin{array}{c} \rho(-X_B - \tau) > \rho(-X_A - \tau) \\
\rho(-X_D - \tau) > \rho(-X_C - \tau) \end{array}\right]$$

also holds. Figure 2 (left panel) shows the entropic PSM values as a function of the target $\tau$ for the second pair of gambles $X_C$ and $X_D$, and for their reflections around zero (right panel). Note that for $X_C$ and $X_D$ the target $\tau^*$ defined in Proposition 10 is approximately 640.

A key issue here, obviously, is target formation; in order for PSP to “explain” such patterns, the targets need to be formed endogenously (i.e., they must be different for the original gambles and their reflected counterparts). We will come back to this point at the end of Section 4. What we have shown here, though, is that if the targets are sufficiently far away from zero (e.g., $\tau > 640$ for the positive gambles and $\tau < -640$ for their reflected counterparts), then the entropic PSM is consistent with these kinds of preference reversals and their reflections. As before, a similar story with different threshold targets would be true for other PSMs.

4 PSPs, ambiguity, and Ellsberg

In this section, we will show that PSPs can be consistent with behavioral choices when the probability distributions of uncertain payoffs are unknown. Ellsberg’s [12] famous experiments provide interesting
insights that decisions made under distributional ambiguity can be inconsistent with the standard paradigm of expected utility and the subjective expected utility theory of Savage [36]. We will show that the PSP model can be extended to accommodate distributional ambiguity and show that it can resolve Ellsberg’s paradoxes across a fairly wide range of targets.

To encompass ambiguity in PSPs, we now confine the probability measure to a family of distributions, \( \mathcal{Q} \). Intuitively speaking, the greater the size of the family \( \mathcal{Q} \), the greater the level of distributional ambiguity. In particular, if the family is a singleton, i.e., \( \mathcal{Q} = \{ \mathbb{P} \} \), then the underlying probability measure is unambiguously specified.

Distributional ambiguity has already been studied in convex risk measures (see Föllmer and Schied, [15]), which are the building blocks of PSMs. Given a law-invariant family of risk measures, \( \mu_{\mathbb{P},k}(X) \), evaluated under the probability measure \( \mathbb{P} \), we can extend this to family of risk measures to encompass distributional ambiguity. For \( k > 0 \), we consider an ambiguity averse risk measure,

\[
\mu_k(X) = \sup_{\mathcal{Q} \in \mathcal{Q}} \mu_{\mathcal{Q},k}(X),
\]

which retains the convexity of the risk measure. For \( k < 0 \), the concave counterpart is given by

\[
\bar{\mu}_k(X) = \inf_{\mathcal{Q} \in \mathcal{Q}} \bar{\mu}_{\mathcal{Q},k}(X),
\]

which corresponds to an ambiguity favoring risk measure. Therefore, we can extend versions of the CVaR, entropic, and homogenized entropic risk measures to encompass distributional ambiguity in straightforward fashion.

**Theorem 3.** Given a law-invariant family of risk measures, \( \mu_{\mathcal{Q},k}(X) \) (normalized, convex for \( k > 0 \), concave for \( k < 0 \)) with \( \mu_{\mathcal{Q},k}(X) \geq \mathbb{E}_{\mathcal{Q}}[-X] \) if \( k > 0 \) and \( \mu_{\mathcal{Q},k}(X) \leq \mathbb{E}_{\mathcal{Q}}[-X] \) if \( k < 0 \), define the corresponding, law-invariant PSM

\[
\rho(X) = \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(X) \leq 0 \}
\]

with

\[
\mu_k(X) = \begin{cases} 
\sup_{\mathcal{Q} \in \mathcal{Q}} \mu_{\mathcal{Q},k}(X) & \text{if } k > 0 \\
\inf_{\mathcal{Q} \in \mathcal{Q}} \mu_{\mathcal{Q},k}(X) & \text{if } k < 0
\end{cases}
\]

Then the following implications hold:

\[
\exists \mathcal{Q} \in \mathcal{Q} : \mathbb{E}_{\mathcal{Q}}[X] < 0 \Rightarrow \rho(X) \leq 0
\]

\[
\exists \mathcal{Q} \in \mathcal{Q} : \mathbb{E}_{\mathcal{Q}}[X] \geq 0 \Rightarrow \rho(X) \geq 0.
\]

Observe that if there exist \( \mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{Q} \) such that \( \mathbb{E}_{\mathcal{Q}_1}[X] < 0 \) and \( \mathbb{E}_{\mathcal{Q}_2}[X] \geq 0 \), then \( X \) is in the neutral set of target premia, i.e., \( \rho(X) = 0 \).

**4.1 Ellsberg’s two-color experiment**

The setup for Ellsberg’s two-color experiment is as follows. Box 1 contains 50 red balls and 50 blue balls. Box 2 contains red and blue balls in unknown proportions. In the first test, subjects are given the following two choices:
• *Gamble A*: Win $100 if ball drawn from Box 1 is red.

• *Gamble B*: Win $100 if ball drawn from Box 2 is red.

In the second test, subjects have to decide between the two choices:

• *Gamble C*: Win $100 if ball drawn from Box 1 is blue.

• *Gamble D*: Win $100 if ball drawn from Box 2 is blue.

In the experimental findings, the majority of subjects are ambiguity averse and strictly prefer gamble A over gamble B and gamble C over gamble D, while a smaller portion are actually ambiguity favoring and strictly prefer gamble B over gamble A and gamble D over gamble C. Ellsberg argues the experimental findings are inconsistent with subjective expected utility theory. The reasoning is as follows: for EUT, individuals who strictly prefer gamble A over gamble B must perceive that in Box 2, red balls are fewer in number than blue ones. In doing so, they would prefer gamble D over gamble C, but this is inconsistent with the experimental findings.

Under Theorem 2, if the corresponding risk measures satisfy the boundedness properties, a PSM on gambles A and C yields non-negative or non-positive values when the target is below or above $50, respectively. For specific PSMs, such as those based on CVaR, entropic and homogenized entropic risk measures, the induced satisficing levels are strictly positive or negative when the target is below or above $50, respectively. In contrast, Theorem 3 implies that for any target between $0 and $100, these PSMs on gambles B and D, which have unknown distributions, are neutral and have satisficing levels valued at zero. Therefore, the preference induced by these PSMs are consistent with the experimental observations.

Clearly, Ellsberg’s paradox can also be resolved by convex or concave risk measures or by worst-case or best-case expected utility under ambiguity depending on whether the individuals are ambiguity averse or favoring (see, e.g., Föllmer and Schied [15] or Gilboa and Schmeidler [18]). The difference here, however, is that PSMs suggest that the ambiguity preferences depend on the aspiration levels of the subjects. We conjecture that this is important in accurately describing decision maker behavior. For instance, if the payoffs in the setup of Ellsberg are increased to $10,000, we may expect subjects who are ambiguity seeking for $100 could possibly switch to ambiguity averse behavior. We have found this to be the case in some informal experiments with hypothetical payoffs (it is costly to implement such an experiment with real payoffs!). It would be interesting to somehow examine this issue with rigorous experimentation.

4.2 Ellsberg’s three-color experiment

In the three color experiment, a box contains 30 red balls and 60 black and yellow balls with unknown proportions. In the first test, subjects choose between the following gambles:

• *Gamble A*: Win $300 if ball drawn from the box is black or yellow.

• *Gamble B*: Win $300 if ball drawn from the box is red or yellow.

In the second test, they have to decide between the two choices:
• **Gamble C:** Win $300 if ball drawn from the box is black.

• **Gamble D:** Win $300 if ball drawn from the box is red.

In gamble A, the probability of winning the $300 prize is 2/3 and the expected payoff is $200. In contrast, the probability of winning the same prize in gamble B ranges from 1/3 to 1. In gamble C, the probability of winning the prize ranges from 0 to 2/3. On the other hand, the probability of winning in gamble D is exactly 1/3 and the expected payoff is $100.

Subjective expected utility theory postulates that individuals who prefer gamble A over gamble B should also prefer gamble C over gamble D. Ellsberg’s experiment reveals, however, that individuals who prefer gamble A over gamble B also tended to prefer gamble D over gamble C; likewise, Ellsberg found that individuals who preferred gamble B over gamble A also tended to prefer gamble C over gamble D.

We present in Table 4 the satisficing levels for all the gambles evaluated using the ambiguity version of the PSMs based on entropic, homogenized entropic and CVaR risk measures. The preferences induced by these PSMs are the same. Gamble A is preferred over gamble B if the target is less than $200 and a reversal of preference occurs when the target exceeds $200. On the other hand, gamble D is preferred over gamble C if the target falls below $100 and a reversal of preference occurs when the target exceeds $100. To reconcile Ellsberg’s experimental findings, some subjects would have to have targets below $100, while others would need targets above $200.

These PSMs state that individuals with targets between $100 to $200 would prefer gamble A over gamble B and gamble C over gamble D. Such pattern of preferences, however, was not mentioned in Ellsberg’s experiment. It is likely that the targets for both experiments may differ since individuals may form their targets only after they have reviewed the different opportunities available. The payoffs in the first test may be perceived to be better than the second test. As such, it does not seem unreasonable for individuals to lower their aspiration levels in the second test. While this explanation seems plausible (see also Simon [37] on how people form targets in the housing market), further experiments on how people form targets are needed. We refer here to a recent literature on this issue; see, for example, Köszegi and Rabin [22, 23].

### 5 Optimizing PSMs and a portfolio choice example

In this section, we show that PSMs can be efficiently optimized over a convex set of random variables, a decision problem that arises in many contexts, including portfolio optimization. This is an advantage for PSMs relative to models of choice under risk that gain descriptive richness by means of, say, a weighting function applied on the cumulative distribution function, as the rank-dependent utility model (RDU) of Quiggin [33] and the cumulative prospect theory (CPT) of Tversky and Kahneman [38]. Indeed, in these models, probability weighting depends on the ranking of states of the world, and the ranking changes as different convex combinations of the random variable are chosen. Consequently, in the context of optimization problems, the weighting of the cumulative distribution function can be difficult to handle and in many applications of RDU or CPT it is ignored, losing the descriptive richness that was the very motivation for the model in the first place.
In contrast, these issues do not arise with PSMs, as we show in this section. We consider the problem of portfolio choice with our model. The use of PSM in a portfolio selection context may be natural in a setting when a manager has high incentives to outperform a pre-specified benchmark, particularly a very aggressive one.

Specifically, given a PSM $\rho$, we consider the problem

$$z^* = \sup \{ \rho(X) : X \in \mathcal{X} \},$$

where $\mathcal{X}$ is the convex hull $\{ \sum_{i=1}^n w_i X_i : \sum_{i=1}^n w_i = 1, w_i \geq 0, \forall i = 1, \ldots, n \}$ of $n$ random variables $X_1, \ldots, X_n$, which give the target premium over a given target $\tau$. From a computational perspective, finding a feasible solution in a convex set is relatively easy compared to finding a feasible solution in a non-convex one. Observe that for $k > 0$, the following set

$$S(k) = \{ X : \rho(X) \geq k, X \in \mathcal{X} \},$$

which can be empty, is convex. If the secured set is nonempty, i.e., $X_{++} \neq \emptyset$, we can efficiently obtain the optimal solution to Problem (8) using the binary search procedure of Brown and Sim [5]. Otherwise, if the secured set is empty, we have $z^* \leq 0$. Moreover, each of the extreme points $X_i$ must either be in the neutral or vulnerable set. If one of them, say $X_j$, is in the neutral set, then if the secured set is empty, $\rho(X_j) = 0$ attains the highest satisficing value over $\mathcal{X}$. Suppose instead that $X_i, i = 1, \ldots, n$, are in the vulnerable set. Then, by quasi-convexity of $\rho$, there exists an extreme point that is optimal. Hence,

$$z^* = \max_{i=1, \ldots, n} \{ \rho(X_i) \},$$

and we can simply enumerate the PSM values for the $n$ extreme points and choose the largest one in this case.

We now consider an asset allocation problem in which the underlying distributions of assets’ returns are not known exactly. Consider $n$ assets with independently distributed returns $V_i, i = 1, \ldots, n$. The exact distribution of $V_i$ is unknown but can be characterized by its support $[v_i, \nu_i]$, i.e., the probability that $V_i$ belongs to $[v_i, \nu_i]$ is one. Also, the mean of $V_i$ is unknown and lies in $[v_i, \nu_i] \subseteq [v_i, \nu_i]$. We then consider the following asset allocation problem:

$$\sup \rho \left( \sum_{i=1}^n w_i V_i - \tau \right)$$

$$\text{s.t. } \sum_{i=1}^n w_i = 1$$

$$w_i \geq 0, \quad i = 1, \ldots, n.$$
We restrict our analysis to the case of an entropic prospective satisficing measure in which the underlying risk measure is given by

\[ \mu_k(X) = \frac{1}{k} \ln \sup_{P \in \mathcal{F}} \mathbb{E}_P [\exp(-Xk)] \]

for \( k > 0 \) and \( k < 0 \), and \( \mathcal{F} \) is the set of probability measures on \((\Omega, \mathcal{F})\) such that \( X \) possesses a feasible distribution (with the given support \([x, \bar{x}]\) and mean in the corresponding interval \([\mu, \bar{\mu}]\)).

**Proposition 11.** Let \( X \) be a random variable with support \([x, \bar{x}]\) and \( \mathcal{F} \) be the set of all admissible distributions of \( X \) such that its mean lies in \([\mu, \bar{\mu}] \subseteq [x, \bar{x}]\). Then

\[
\sup_{P \in \mathcal{F}} \mathbb{E}_P [\exp(-aX)] = \begin{cases} 
    \bar{p} \exp(a\bar{x}) + \bar{q} \exp(a\mu) & \text{if } a \geq 0 \\
    \bar{p} \exp(a\bar{x}) + \bar{q} \exp(a\mu) & \text{otherwise},
\end{cases}
\]

where \( \bar{p} = (\bar{x} - \mu)/(|x - \bar{x}|) \), \( \bar{q} = 1 - \bar{p} \), \( \bar{p} = (\bar{x} - \mu)/(|x - \bar{x}|) \) and \( \bar{q} = 1 - \bar{p} \).

Given a target \( \tau \), Proposition 11 enables us to compute the EPSM under the given distributional ambiguity. Suppose \( \rho(V_i - \tau) < 0 \) for all \( i \), then all assets are in the vulnerable set and concentration is favored. Hence, it is optimal to invest in the asset with the highest \( \rho(V_i - \tau) \). Otherwise, we solve the following optimization problem

\[
\sup \ k \\
\text{s.t.} \sum_{i=1}^{n} \frac{1}{k} \ln(p_i \exp(-w_i k \nu_i) + q_i \exp(-w_i k \nu_i)) \leq -\tau \\
\sum_{i=1}^{n} w_i = 1 \\
k > 0, w_i \geq 0, \quad i = 1, \ldots, n,
\]

where \( p_i = (\nu_i - \bar{\nu}_i)/(\nu_i - \bar{\nu}_i), q_i = 1 - p_i \). This is equivalent to the following problem:

\[
\inf \ a \\
\text{s.t.} \sum_{i=1}^{n} a \ln(p_i, \exp(-w_i \nu_i/a) + q_i \exp(-w_i \nu_i/a)) \leq -\tau \\
\sum_{i=1}^{n} w_i = 1 \\
a > 0, w_i \geq 0, \quad i = 1, \ldots, n,
\]

which is a convex optimization problem, and hence can be solved, in high dimension, efficiently using interior point methods.\(^8\)

We now present a numerical example based on the information presented in Table 5. Note that the asset returns are defined in such a way that asset 1 is a risk-free asset that pays 2% in all states of the world. Assets 2 to 6 are risky assets, with asset 2 being the one with lowest downside and upside and asset 6 being the one the highest downside and upside. We solve the optimal asset allocation for various

\(^8\)For this example, it is convenient to use a solver that can explicitly handle the “exponential cone”; here we use the software package ROME [17] to solve our example problem.
targets as presented in Table 6. The lowest target corresponds to the risk-free asset. In this case, the investor can reach the target for sure by investing in asset 1. As the target increases, the risk-free asset 1 becomes less attractive, because it fails to attain the target with certainty. The investor puts some of their wealth into the risky assets. If the target becomes very high, i.e., the investor is very ambitious, then they only hold asset 6, the asset with the highest upside potential and a positive probability to be above the target.

This example highlights the intuitive idea that if the investor possesses a high target return, then they will be willing to take more risk. This pattern is similar to that observed in mutual fund managers during the technology bubble of the 1990’s and discussed in Dass et al. [7]. Managers with high contractual incentives to rank at the top (i.e., those with a high target) adopted the risky and aggressive strategy to not invest in bubble stocks, as this was the only way to outperform the market. Such a strategy also carried with it a high probability of ranking at the bottom if the bubble continued. In contrast, mutual fund managers with a high incentive to follow the benchmark (i.e., those mainly concerned about not ranking at the bottom, thus with a low target) adopted the less risky strategy to follow the bubble (herding), which yields a small probability of ranking at the bottom. While the observation of Dass et al. [7] suggests that fund managers’ decisions may be target-based, we do not want to overemphasize here the ability of PSPs to explain real portfolio choices. Addressing portfolio selection and asset pricing implications of PSPs seems like an interesting subject for future research.

6 Conclusion

In this paper, we have developed prospective satisficing preferences, a target-based model of risky choice. Our model has several interesting features. First, we obtained a representation theorem that has practical relevance, since it links our framework to a more standard definition of a risk measure. In addition, under mild assumptions, the PSP model preserves first-order stochastic dominance over all premia, second-order stochastic dominance over secured premia, and risk-seeking second-order stochastic dominance over vulnerable premia. Moreover, PSPs can address several observed violations of expected utility theory, thus also displaying considerable descriptive power. Finally, the model is amenable to computationally efficient optimization.

There are a number of interesting directions for future research. One direction is to consider the use of the PSP model in various applications, such as portfolio choice or environments involving competition. It would be interesting to see what implications the model has for market portfolios and equilibrium behavior. Second, it would be useful to have more detailed experimental work that tests the implications of the model and examines elicitation of targets.

Finally, and related to this, is further investigation of the descriptive power of the PSP model. There is a large body of empirical evidence that suggests that performance relative to a target payoff is a critical factor driving the decision making of many individuals. We have found that the PSP model can also explain some recently observed “puzzles” in decision theory, in addition to the classical ones discussed earlier. For instance, Machina [27] recently pointed out that rank-dependent preferences that resolve the classical Ellsberg example can still struggle with other “Ellsberg-like” examples due to a required separability property. It turns out PSPs do not impose such separability and therefore
can address these instances. As another example, Wu and Markle [40] have recently pointed out that people demonstrate systematic violations of double matching, which is necessary for the representation of cumulative prospect theory and states that if a decision maker is indifferent between both the tail gains and tail losses of two gambles, then they must be indifferent between the composition of these two tails as well. One can verify that PSPs need not obey this stringent requirement of gain-loss separability either. What we find appealing is that the PSP model is fairly simple to motivate and not done with any particular descriptive intent in mind, yet it still can readily accommodate a fairly wide array of phenomena that seem to be present in the decision making of many individuals.

References


Appendix

Proofs

Proof of Proposition 1

Let $\succeq$ be a prospective satisficing preference relation. Property 1 implies the existence of an upper semi-continuous function $\tilde{\rho} : \mathcal{X} \to \mathbb{R}$ such that $X \succeq Y$ if and only if $\tilde{\rho}(X) \geq \tilde{\rho}(Y)$ (Theorem 4, Bosi and Mehta [4]).

Monotonicity for $\tilde{\rho}$ follows directly from Property 2 (that is, monotonicity of $\succeq$).

Property 3 implies that $\tilde{\rho}$ is constant and maximal on $\{X \in \mathcal{X} : X \geq 0\}$ and constant and minimal on $\{X \in \mathcal{X} : X < 0\}$. Assume that $\tilde{\rho}(X) = \tilde{\rho}_u$ for all $X \geq 0$ and $\tilde{\rho}(X) = \tilde{\rho}_l$ for all $X < 0$, then Property 3 also implies $\tilde{\rho}(Y) \in [\tilde{\rho}_l, \tilde{\rho}_u]$ for all $Y \in \mathcal{X}$.

Property 4 implies that $\tilde{\rho}$ is constant on $\mathcal{X}_0$. Assume $\tilde{\rho}(X) = \tilde{\rho}_0$ for all $X \in \mathcal{X}_0$, then Property 4 also implies that $\tilde{\rho}(X) < \tilde{\rho}_0$ for all $X \in \mathcal{X}_-$ and $\tilde{\rho}(X) > \tilde{\rho}_0$ for all $X \in \mathcal{X}_+$. From this it follows that $\tilde{\rho}_u > \tilde{\rho}_0 > \tilde{\rho}_l$.

Property 4(i) implies quasi-concavity\(^9\) for $\tilde{\rho}$ on $\mathcal{X}_+$ and Property 4(ii) implies quasi-convexity for $\tilde{\rho}$ on $\mathcal{X}_-$. Let $\rho(X) = g(\tilde{\rho}(X)) - g(\tilde{\rho}_0)$ where $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty, \infty\}$ is defined as follows:

$$
g(x) = \begin{cases} 
-\infty & x = \tilde{\rho}_l \\
\frac{1}{\tilde{\rho}_u - x} & x \in (\tilde{\rho}_l, \tilde{\rho}_0) \\
\frac{1}{\tilde{\rho}_u - \tilde{\rho}_0} - \frac{1}{\tilde{\rho}_u - \tilde{\rho}_0} & x = \tilde{\rho}_u
\end{cases}
$$

Since $g$ is strictly increasing, it preserves the ordering of $\tilde{\rho}$ and $\rho$ satisfies properties (i), (ii) and (iii)(a). Moreover, since $g$ is continuous, $\rho$ is also upper semi-continuous, quasi-concave on $\mathcal{X}_+$ and quasi-convex on $\mathcal{X}_-$. This proves the existence of $\rho$ with properties (i)-(iv).

On the other hand, it is straightforward to show that a upper semi-continuous function $\rho : \mathcal{X} \to \mathbb{R} \cap \{-\infty, \infty\}$ with properties (i)-(iv) defines a prospective satisficing preference relation $\succeq$ with $X \succeq Y$ if and only if $\rho(X) \geq \rho(Y)$. Note that when $\rho$ is upper semi-continuous, then it follows from Bosi and Mehta [4] that $\succeq$ satisfies Properties (i) and (ii) in Property 1.

\(^9\)Assume wlog that $X \succeq Y$. Then $\lambda X + (1 - \lambda)Y \succeq Y$, so $\rho(\lambda X + (1 - \lambda)Y) \geq \rho(Y) = \min(\rho(X), \rho(Y))$; an analogous argument follows for quasi-convexity on $\mathcal{X}_-$.  


Proof of Proposition 2

Clearly, all convex functions are also quasi-convex. It suffices to show that a quasi-convex function that satisfies translation invariance is always convex. We have:

\[ \mu(\lambda X + (1 - \lambda)Y) - (\lambda \mu(X) + (1 - \lambda)\mu(Y)) = \mu(\lambda(X + \mu(X)) + (1 - \lambda)(Y + \mu(Y))) \leq \max\{\mu(X + \mu(X)), \mu(Y + \mu(Y))\} = \max\{0, 0\} = 0. \]

Hence,

\[ \mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y). \]

That a quasi-concave function that satisfies translation invariance is always concave follows by an analogous argument.

Proof of Theorem 1

Suppose \( \rho \) takes the form (5) and \( \mu_k \) is the family of risk measures described in Theorem 1; we will show that \( \rho \) is a prospective satisficing measure.

1. **Upper semi-continuity:**

   Upper semi-continuity for \( \rho \) is equivalent to \( \{X \in \mathcal{X} : \rho(X) \geq k\} \) being closed for all \( k \). Let \( k \in (-\infty, \infty) \setminus \{0\} \) and take a sequence \( (X_n)_n \) in \( \{X \in \mathcal{X} : \rho(X) \geq k\} \) such that \( X_n \to X \) as \( n \to \infty \). Since \( \rho(X_n) \geq k \), then \( \mu_k(X_n) \leq 0 \). Therefore, the sequence \( (X_n)_n \) belongs to the acceptance set \( \mathcal{A}_{\mu_k} \). Since \( \mathcal{A}_{\mu_k} \) is closed, then \( X \in \mathcal{A}_{\mu_k} \), i.e., \( \mu_k(X) \leq 0 \). This implies \( \rho(X) \geq k \), i.e., \( X \in \{X \in \mathcal{X} : \rho(X) \geq k\} \). This proves that \( \{X \in \mathcal{X} : \rho(X) \geq k\} \) is closed for all \( k \), and thus \( \rho \) is upper semi-continuous.

2. **Monotonicity:**

   Follows clearly from monotonicity of the underlying risk measures.

3. **Satisficing behavior:**

   (a) **Attainment content:**

      Note that if \( X \geq 0 \), then monotonicity for \( \mu_k \) implies \( \mu_k(X) \leq \mu_k(0) = 0 \), for all \( k \in \mathbb{R} \setminus \{0\} \). Hence, \( \rho(X) = \infty \).

   (b) **Non-attainment apathy:**

      If \( X < 0 \), there exists \( \epsilon < 0 \) such that \( X \leq \epsilon \). Hence, monotonicity for \( \mu_k \) implies \( \mu_k(X) \geq \mu_k(\epsilon) = -\epsilon > 0 \) for all \( k \in \mathbb{R} \setminus \{0\} \); therefore, \( \{k : \mu_k(X) \leq 0\} = \emptyset \), so \( \rho(X) = \sup\{\} = -\infty \) (by our convention that \( \sup\emptyset = -\infty \)).

4. **Prospective behavior:**

   (a) **Superiority of secured target premia and inferiority of vulnerable target premia:**

      Follows by definition of the secured, vulnerable, and neutral sets in (2).
(b) **Quasi-concavity over secured target premia:**

Let \( X, Y \in \mathcal{X}_+ \) and \( k^* = \min \{\rho(X), \rho(Y)\} > 0 \). Note that \( \mu_k(X) \leq 0 \) and \( \mu_k(Y) \leq 0 \) for all \( k \in (0, k^*) \). Then, using convexity of \( \mu_k \) on \( k > 0 \), we have

\[
\rho(\lambda X + (1 - \lambda)Y) = \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(\lambda X + (1 - \lambda)Y) \leq 0 \} \\
\geq \sup \{ k \in (0, \infty) : \mu_k(\lambda X + (1 - \lambda)Y) \leq 0 \} \\
\geq \sup \{ k \in (0, \infty) : \lambda \mu_k(X) + (1 - \lambda)\mu_k(Y) \leq 0 \} \\
\geq k^* \\
= \min \{\rho(X), \rho(Y)\}.
\]

(c) **Quasi-convexity over vulnerable target premia:**

Let \( X, Y \in \mathcal{X}_- \) and \( k^* = \max \{\rho(X), \rho(Y)\} < 0 \). Note that \( \mu_k(X), \mu_k(Y) > 0 \) for all \( k > k^* \). Hence, for all \( k \in (k^*, 0) \),

\[
\mu_k(\lambda X + (1 - \lambda)Y) \geq \lambda \mu_k(X) + (1 - \lambda)\mu_k(Y) > 0.
\]

Since \( \mu_k \) is nondecreasing in \( k \), the above inequality also holds for \( k > k^* \). Therefore, we have

\[
\rho(\lambda X + (1 - \lambda)Y) = \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(\lambda X + (1 - \lambda)Y) \leq 0 \} \\
= \sup \{ k \in (-\infty, k^*] : \mu_k(\lambda X + (1 - \lambda)Y) \leq 0 \} \\
\leq k^* \\
= \max \{\rho(X), \rho(Y)\}.
\]

For the other direction, let \( \mu_k \) take the form (6) in which \( \rho \) is a prospective satisficing measure. By monotonicity of \( \rho \), it is clear that \( \mu_k \) is nondecreasing on \( k \). To verify that \( \mu_k \) is a risk measure with a closed acceptance set, we note the following:

1. **Closed acceptance set:**

   We show that \( \mu_k(X) \leq 0 \) is equivalent to \( \rho(X) \geq k \). One direction is trivial, i.e., when \( \rho(X) \geq k \) then \( \mu_k(X) \leq 0 \). For the other direction, we note that upper semi-continuity for \( \rho \) implies upper semi-continuity for \( a \rightarrow \rho(X + a) \), for all \( X \in \mathcal{X} \). Moreover, since \( a \rightarrow \rho(a + X) \) is also increasing due to the monotonicity of \( \rho \), then it is also right-continuous and the limit of Problem (6) is achievable. It follows that when \( \mu_k(X) \leq 0 \), then there exists an \( a \leq 0 \) such that \( \rho(a + X) \geq k \). Due to monotonicity of \( \rho \) we also have \( \rho(X) \geq k \). We have thus showed:

   \[
   A_{\mu_k} = \{ X \in \mathcal{X} : \mu_k(X) \leq 0 \} = \{ X \in \mathcal{X} : \rho(X) \geq k \}.
   \]

   Since \( \rho \) is upper semi-continuous, \( \{ X \in \mathcal{X} : \rho(X) \geq k \} \) is closed and thus so is \( A_{\mu_k} \).

2. **Monotonicity:**

   Clear.
3. Translation invariance:

For all $c \in \mathbb{R}$,

$$
\mu_k(X + c) = \inf\{a : \rho(X + c + a) \geq k\} = \inf\{a - c : \rho(X + a) \geq k\} = \mu_k(X) - c.
$$

4. Convexity on $k > 0$:

Given $X, Y \in \mathcal{X}$, notice that, by monotonicity of $\rho$ and the definition of $\mu_k$, we have for all $\epsilon > 0$,

$$
\rho(X + \mu_k(X) + \epsilon) \geq k
$$

and

$$
\rho(Y + \mu_k(Y) + \epsilon) \geq k.
$$

Since $k > 0$, we have $X + \mu_k(X) + \epsilon, Y + \mu_k(Y) + \epsilon \in \mathcal{X}_{++}$. For every $\lambda \in [0, 1]$, define

$$
a_\lambda \triangleq \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y).
$$

Then, for all $\epsilon > 0$,

$$
\rho(\lambda X + (1 - \lambda) Y + a_\lambda + \epsilon) = \rho(\lambda (X + \mu_k(X)) + \epsilon + (1 - \lambda)(Y + \mu_k(Y) + \epsilon))
$$

$$
\geq \min\{\rho(X + \mu_k(X) + \epsilon), \rho(Y + \mu_k(Y) + \epsilon)\}
$$

$$
\geq k > 0.
$$

Then

$$
\mu_k(\lambda X + (1 - \lambda) Y) = \inf\{a : \rho(\lambda X + (1 - \lambda) Y + a) \geq k\}
$$

$$
\leq a_\lambda
$$

$$
= \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y).
$$

5. Concavity on $k < 0$:

Since $\mu_k(X) = \inf\{a : \rho(X + a) \geq k\}$, it follows that $\rho(X + \mu_k(X) + a) < k < 0$ and $\rho(Y + \mu_k(Y) + a) < k < 0$ for all $a < 0$. Therefore, for all $a < 0$, $X + \mu_k(X) + a \in \mathcal{X}_{-}$, and $Y + \mu_k(Y) + a \in \mathcal{X}_{-}$; hence,

$$
\rho(\lambda (X + \mu_k(X)) + (1 - \lambda)(Y + \mu_k(Y)) + a) \leq \max\{\rho(X + \mu_k(X) + a), \rho(Y + \mu_k(Y) + a)\} < k
$$

for all $\lambda \in [0, 1]$. Therefore,

$$
\mu_k(\lambda (X + \mu_k(X)) + (1 - \lambda)(Y + \mu_k(Y)))
$$

$$
= \inf\{a : \rho(\lambda (X + \mu_k(X)) + (1 - \lambda)(Y + \mu_k(Y)) + a) \geq k\}
$$

$$
= \inf\{a : \rho(\lambda (X + \mu_k(X)) + (1 - \lambda)(Y + \mu_k(Y)) + a) \geq k, a \geq 0\}
$$

$$
\geq 0.
$$

Concavity then follows from the translation invariance property of $\mu_k$. 

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Finally, we need to show that
\[ \rho(X) = \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(X) \leq 0 \}. \]

We have seen in (i) above that the limit of Problem (6) is achievable. Therefore,
\[
\sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(X) \leq 0 \}
= \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \exists a \leq 0 \text{ s.t. } \rho(X + a) \geq k \}
= \sup \{ \rho(X + a) : a \leq 0 \}
= \rho(X),
\]
which completes the proof.

Proof of Proposition 3

It is straightforward to verify the concavity and nondecreasing properties of \( \bar{\mu}_k \). Moreover, for all \( k > 0 \),
\[
0 = \mu_k \left( \frac{1}{2}(X - X) \right) \leq \frac{1}{2} \mu_k(X) + \frac{1}{2} \mu_k(-X).
\]
Hence, \( \bar{\mu}_{-k}(X) \leq \mu_k(X) \) for all \( k > 0 \). Therefore, for all \( s < 0, t > 0 \),
\[
\bar{\mu}_s(X) \leq \lim_{k \uparrow 0} \bar{\mu}_k(X) \leq \lim_{k \uparrow 0} \mu_k(X) \leq \mu_t(X).
\]

Proof of Proposition 4

Let \( X \in \mathcal{X} \) such that \( k \mapsto \mu_k(X) \) is strictly increasing. We have
\[
\rho(-X) = \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(-X) \leq 0 \}
= \sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_{-k}(X) \geq 0 \}
= -\inf \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(X) \geq 0 \}
= -\sup \{ k \in (-\infty, \infty) \setminus \{0\} : \mu_k(X) \leq 0 \}
= -\rho(X).
\]
The first and the last equalities follow from Theorem 1, the second equality is given by the symmetric properties of the family of risk measures \( \{ \mu_k : k \in \mathbb{R} \setminus \{0\} \} \), and the fourth equality holds since \( k \mapsto \mu_k(X) \) is strictly increasing.

If \( X \) is deterministic, then \( k \mapsto \mu_k(X) \) is also constant and the argument above does not hold. In this case, however, if \( X > 0 \), then \( \rho(X) = \infty \) and \( \rho(-X) = -\infty \). If \( X < 0 \), the opposite holds. Thus, if \( X \) is deterministic, \( X \neq 0 \), we also have \( \rho(-X) = -\rho(X) \). In contrast, if \( X = 0 \), then \( \infty = \rho(-X) \neq -\rho(X) = -\infty \).
Proof of Proposition 5

Suppose that $X \geq (1) Y$. Then $E[u(X)] \geq E[u(Y)]$ for all $u$ nondecreasing and the inequality is strict for at least one such $u$. Since $u(x)$ is nondecreasing if and only if $-u(-x)$ is also nondecreasing, we have $-E[u(-X)] \geq -E[u(-Y)]$, or, equivalently, $E[u(-X)] \leq E[u(-Y)]$ for $u$ nondecreasing and the inequality is strict for at least one such $u$. This implies that $-Y \geq (1) -X$. Therefore,

$$
\hat{\mu}(X) = -\mu(-X) \leq -\mu(-Y) = \hat{\mu}(Y).
$$

For SSD, we observe that a function $u(x)$ is nondecreasing and concave if and only if $-u(-x)$ is nondecreasing and convex. Hence, $X \geq (2) Y$ if and only if $-Y \geq (2) -X$. The result now follows in similar fashion to above.

Proof of Proposition 6

Note that if $X \geq (1) Y$, then $\mu_k(X) \leq \mu_k(Y)$ for all $k \in \mathbb{R} \setminus \{0\}$, since $\mu_k$ preserves FSD. By the definition of $\rho$, it follows immediately that $\rho(X) \geq \rho(Y)$, i.e., $\rho$ also preserves FSD.

For the next claim, note that $Y \in \mathcal{X}_{++}$ implies that $\rho(Y) > 0$. Since $X \geq (2) Y$ and $\mu_{\rho(Y)}$ preserves SSD, we have

$$
\mu_{\rho(Y)}(X) \leq \mu_{\rho(Y)}(Y) \leq 0.
$$

Therefore, $\rho(X) \geq \rho(Y)$.

Likewise, $Y \in \mathcal{X}_{--}$ implies that $\rho(Y) < 0$. Since $X \geq (2) Y$ and $\mu_{\rho(Y)}$ preserves RSSD, we have

$$
\mu_{\rho(Y)}(X) \leq \mu_{\rho(Y)}(Y) \leq 0.
$$

Therefore, $\rho(X) \geq \rho(Y)$.

Proof of Proposition 7

First, it is easy to see that law-invariance of the PSM implies law-invariance of the underlying family of risk measures (see Equation (6)). Föllmer and Schied [15] show that on atomless probability spaces any law-invariant risk measure preserves FSD, and any convex, law-invariant risk measure preserves SSD; the claim now follows by Propositions 5 and 6.

Proof of Theorem 2

The boundedness properties for the family $\{\mu_k : k \in (-\infty, \infty) \setminus \{0\}\}$ imply $\mu_k(X) \geq E[-X]$ for $k > 0$ and $\mu_k(X) \leq E[-X]$ for $k < 0$. It follows that when $E[X] < 0$, then for all $k > 0$

$$
\mu_k(X) \geq E[-X] > 0.
$$

Hence, it follows from Theorem 1 that $\rho(X) \leq 0$. Likewise, if $E[X] \geq 0$, then for all $k < 0$,

$$
\mu_k(X) \leq E[-X] \leq 0.
$$

Again, following Theorem 1, we have $\rho(X) \geq 0$. 

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Finally, for the homogenized entropic risk measure, we have
\[ \mu_k(X) = -\mu_{-k}(-X) \leq \mathbb{E}[-X]. \]
Henceforth, we assume \( k > 0 \). For the CVaR risk measure, we have, by Jensen’s inequality
\[ \mu_k(X) = \frac{1}{k} \ln \mathbb{E}[\exp(-kX)] \geq \frac{1}{k} \ln \mathbb{E}[\exp(-kX)] = -\mathbb{E}[X]. \]
For the CVaR risk measure, we have
\[
\begin{align*}
\mu_k(X) &= \inf_{\nu \in \mathbb{R}} \{ \nu + e^k \mathbb{E}[(X - \nu)^+] \} \\
&\geq \inf_{\nu \in \mathbb{R}} \{ \nu + \mathbb{E}[(X - \nu)^+] \} \\
&\geq \inf_{\nu \in \mathbb{R}} \{ \nu + \mathbb{E}[-X - \nu] \} \\
&= \mathbb{E}[-X].
\end{align*}
\]
Finally, for the homogenized entropic risk measure, we have
\[
\begin{align*}
\mu_k(X) &= \inf_{a > 0} \{ a \ln (\mathbb{E}[\exp(-X/a)]) + ak \} \\
&\geq \inf_{a > 0} \{ a \ln (\mathbb{E}[\exp(-X/a)]) \} \\
&\geq \inf_{a > 0} \{ a \ln (\exp(-\mathbb{E}[X]/a)) \} \\
&= \mathbb{E}[-X].
\end{align*}
\]

**Proof of Proposition 8**

Since these are symmetric families of risk measures, it suffices to show that \( \mu_k(X) \geq \mathbb{E}[-X] \) for \( k > 0 \), which implies that for \( k < 0 \)
\[ \mu_k(X) = -\mu_{-k}(-X) \leq \mathbb{E}[-X]. \]

**Proof of Proposition 9**

First, it is clear that \( \rho(X_A - \tau) = \infty \) and \( \rho(X_B - \tau) < \infty \) for any \( \tau \in (0, y] \). We thus focus on comparing \( X_C \) to \( X_D \) over the range \( \tau \in (0, y] \).

There is a one-to-one mapping between target levels and satisficing levels. In particular, for a particular satisficing level \( \rho \), let \( \tau(X_C, \rho) \) and \( \tau(X_D, \rho) \) be the corresponding target levels that induce the satisficing level \( \rho \) for gambles \( X_C \) and \( X_D \), respectively. We have
\[
\begin{align*}
\tau(X_C, \rho) &= \frac{1}{\rho} \log \left[ 1 + p(e^{-x_\rho} - 1) \right] \\
\tau(X_D, \rho) &= \frac{1}{\rho} \log \left[ 1 + q(e^{-y_\rho} - 1) \right].
\end{align*}
\]
Note that \( \tau(X_C, \rho) \) and \( \tau(X_D, \rho) \) are both decreasing and continuous in \( \rho \). We will compare these target functions as \( \rho \) varies and will show that there exists a unique \( \rho^* > 0 \) such that \( \tau(X_C, \rho^*) = \tau(X_D, \rho^*) \), and that \( \tau(X_D, \rho) > \tau(X_C, \rho) \) for all \( \rho > \rho^* \), and \( \tau(X_D, \rho) < \tau(X_C, \rho) \) for all \( \rho < \rho^* \). This shows that \( \rho(X_C - \tau) > \rho(X_D - \tau) \) if and only if \( \tau > \tau(X_C, \rho^*) = \tau(X_D, \rho^*) \).

First, consider \( \rho < 0 \). Over this range, we have
\[
\begin{align*}
\tau(X_C, \rho) > \tau(X_D, \rho) &\iff -\frac{1}{\rho} \log \left[ 1 + p(e^{-x_\rho} - 1) \right] > -\frac{1}{\rho} \log \left[ 1 + q(e^{-y_\rho} - 1) \right] \\
&\iff p(e^{-x_\rho} - 1) - q(e^{-y_\rho} - 1) > 0.
\end{align*}
\]
Let $g(\rho) = p(e^{-x^p} - 1) - q(e^{-y^p} - 1)$, the left hand side of the latter inequality. Over $\rho < 0$, we have

$$g'(\rho) = -pxe^{-x^p} + qye^{-y^p} < qy(e^{-y^p} - e^{-x^p}) < 0,$$

where in the first line we use the fact that $px > qy$ and in the second line we use $\rho < 0$ and $y > x$. In addition,

$$\lim_{\rho \to 0} g(\rho) = 0 \quad \lim_{\rho \to -\infty} g(\rho) = +\infty.$$ 

In sum, $g(\rho)$ is a strictly decreasing function from $+\infty$ to 0 as $\rho \uparrow 0$ and therefore must be strictly positive over the range, which implies that $g(\rho) > 0$ over $\rho < 0$, and thus $\tau(X_C, \rho) > \tau(X_D, \rho)$ over this range.

For $\rho = 0$, the target levels reduce to the expected values; thus, $\tau(X_C, 0) = px > qy = \tau(X_D, 0)$.

Finally, consider $\rho > 0$. Similar to the first case, we have over this range

$$\tau(X_C, \rho) > \tau(X_D, \rho) \iff p(e^{-x^p} - 1) - q(e^{-y^p} - 1) < 0.$$ 

Let $h(\rho) = p(e^{-x^p} - 1) - q(e^{-y^p} - 1)$, the left hand side of the latter inequality. We have $\lim_{\rho \to 0} h(\rho) = 0$ and $\lim_{\rho \to -\infty} h(\rho) = q - p > 0$. Moreover, $h'(\rho) = -pxe^{-x^p} + qye^{-y^p}$, so

$$h'(\rho) \leq 0 \iff \rho \leq \left(\frac{1}{x - y}\right) \log \left[\frac{px}{qy}\right] = \hat{\rho} > 0.$$ 

Thus, $h(\rho)$ over $\rho \geq 0$ has a left limit of zero, a right limit of the positive value $q - p$, and is nonincreasing for $\rho \leq \hat{\rho}$ and increasing otherwise. This implies that there exists a unique $\rho^* \geq \hat{\rho} > 0$ when $h(\rho)$ crosses zero. Note furthermore that $h(\rho^*) = 0$ is equivalent to $\tau(X_C, \rho^*) = \tau(X_D, \rho^*)$, i.e., $\mu_{\rho^*}(X_C) = \mu_{\rho^*}(X_D)$. Also, since $\rho^* > 0$, we must have $\tau(X_C, \rho^*) = \tau(X_D, \rho^*) < \mu(X_D) = qy$ as claimed.

In summary, we have shown that there is a single target level $\tau^*$ with the desired construction such that the satisficing levels of $X_C$ and $X_D$ coincide at $\tau^*$; below $\tau^*$, $X_D$ is preferred to $X_C$ and vice versa for above $\tau^*$. This completes the proof. 

\[\square\]

**Proof of Proposition 10**

First, notice that for any $\tau < y$, $\rho(X_A - \tau) = \infty$ and $\rho(X_B - \tau) < \infty$, so $\rho(X_A - \tau) > \rho(X_B - \tau)$. Now consider $X_C$ and $X_D$. For $X_D$, any target $\tau < y$ induces a corresponding value $\hat{\rho}(\tau) = \rho(X_D - \tau)$, where using the representation theorem for $\rho$, we find

$$\frac{1}{\hat{\rho}(\tau)} \log \left[(1 - q)e^{-\hat{\rho}(\tau)y} + q\right] = -\tau \iff \tau = -\rho(\tau)(X_D)$$

must hold. Notice that $\hat{\rho}(\tau)$ is monotonically decreasing on $\tau \in (0, y)$. In order to have $\rho(X_C - \tau) > \rho(X_D - \tau)$, we must have $\mu_{\hat{\rho}(\tau)}(X_C - \tau) < 0$. Considering separately the two cases whether $\hat{\rho}(\tau) > 0$ or $\hat{\rho}(\tau) \leq 0$, we find that in either case, this is equivalent to the condition

$$\frac{1}{\hat{\rho}(\tau)} \log \left[pe^{-\hat{\rho}(\tau)x} + (1 - p)\right] < -y \iff \mu_{\hat{\rho}(\tau)}(X_B - y) < 0.$$
Therefore, we can choose \( \tau \) small enough such that \( \hat{\rho}(\tau) \leq \rho(X_B - y) \) holds, which leads to the threshold value \( \tau^* \) in the result. Notice that \( x > y > 0 \) and \( p > 0 \) imply that \( \rho^* = \rho(X_B - y) > 0 \); this in conjunction with \( q < 1 \) implies that \( \tau^* = -\mu_\rho(X_D) < y \).

For \( y > \tau > \tau^* \), we have \( \hat{\rho}(\tau) < \rho^* \), so \( \rho(X_C - \tau) > \rho(X_D - \tau) \) and \( \rho(X_A - \tau) > \rho(X_B - \tau) \) over the range \((\tau^*, y)\), as required. \( \square \)

**Proof of Theorem 3**

Suppose there exists a \( Q^* \in Q \) such that \( E_{Q^*}[X] < 0 \). From Theorem 2, we have, for \( k > 0 \),

\[
\mu_k(X) = \sup_{Q \in \mathcal{Q}} \mu_{Q,k}(X) \geq \mu_{Q^*,k}(X) \geq E_{Q^*}[X] > 0.
\]

Hence, \( \rho(X) \leq 0 \). Likewise, suppose there exists a \( Q^* \in Q \) such that \( E_{Q^*}[X] \geq 0 \); we have, for \( k < 0 \),

\[
\mu_k(X) = \inf_{Q \in \mathcal{Q}} \mu_{Q,k}(X) \leq \mu_{Q^*,k}(X) \leq E_{Q^*}[X] \leq 0,
\]

and again invoking the representation theorem for PSMs, it must be that \( \rho(X) \geq 0 \). \( \square \)

**Proof of Proposition 11**

The worst case expectation can be obtained by solving the following optimization problem:

\[
\begin{align*}
\sup_f \quad & E_f[\exp(-aX)] \\
\text{s.t.} \quad & \mu \leq E_f[X] \leq \bar{\mu}, \\
& E_f[1] = 1 \\
& f(x) \geq 0, \quad \forall x \in [\underline{x}, \bar{x}].
\end{align*}
\]

We can consider \( f \) to be an infinite dimensional vector indexed by \( x \in [\underline{x}, \bar{x}] \). By weak duality, the upper bound to the above problem can be obtained by

\[
\begin{align*}
& \inf_{r,s,t} \left\{ r + \bar{\mu}s - \underline{\mu}t : r + xs - xt \geq \exp(-ax) \forall x \in [\underline{x}, \bar{x}], s, t \geq 0 \right\} \\
= & \inf_{r,s,t} \left\{ r + \bar{\mu}s - \underline{\mu}t : r \geq \max_{x \in [\underline{x}, \bar{x}]} \{ \exp(ax) - xs + xt \}, s, t \geq 0 \right\} \\
= & \inf_{r,s,t} \left\{ r + \bar{\mu}s - \underline{\mu}t : r \geq \max\{\exp(-a\bar{x}) - \bar{x}s + \bar{\mu}t, \exp(-a\underline{x}) - \bar{x}s + \bar{\mu}t, \exp(-a\bar{x}) - \underline{x}s + \underline{\mu}t, \exp(-a\underline{x}) - \underline{x}s + \underline{\mu}t \}, s, t \geq 0 \right\} \\
= & \inf_{s,t} \left\{ \max\{\exp(-a\bar{x}) - \bar{x}s + \bar{\mu}t, \exp(-a\underline{x}) - \bar{x}s + \bar{\mu}t, \exp(-a\bar{x}) - \underline{x}s + \underline{\mu}t, \exp(-a\underline{x}) - \underline{x}s + \underline{\mu}t \} : s, t \geq 0 \right\}.
\end{align*}
\]

By inspection, when \( a \geq 0 \), strong duality is achieved by a two point distribution with \( P(X = \underline{x}) = \underline{p} \) and \( P(X = \bar{x}) = \bar{p} \) and dual variables \( s = 0, t = (\exp(-a\bar{x}) - \exp(-a\bar{x}))/\bar{\mu} \geq 0 \). Likewise, when \( a < 0 \), strong duality is achieved by a two point distribution with \( P(X = \underline{x}) = \underline{p} \) and \( P(X = \bar{x}) = \bar{p} \) and dual variables \( s = (\exp(-a\bar{x}) - \exp(-a\bar{x}))/\bar{\mu} \geq 0, t = 0 \). \( \square \)
**Tables**

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Gamble A</th>
<th>Gamble B</th>
<th>Gamble C</th>
<th>Gamble D</th>
</tr>
</thead>
<tbody>
<tr>
<td>55,000</td>
<td>$\infty$</td>
<td>$83.7 \times 10^{-6}$</td>
<td>$1.90 \times 10^{-6}$</td>
<td>0</td>
</tr>
<tr>
<td>250,000</td>
<td>$\infty$</td>
<td>$18.4 \times 10^{-6}$</td>
<td>0</td>
<td>$-8.36 \times 10^{-6}$</td>
</tr>
<tr>
<td>500,000</td>
<td>$\infty$</td>
<td>$4.80 \times 10^{-6}$</td>
<td>$-0.60 \times 10^{-6}$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>695,000</td>
<td>$-\infty$</td>
<td>0</td>
<td>$-0.92 \times 10^{-6}$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>2,000,000</td>
<td>$-\infty$</td>
<td>$-4.60 \times 10^{-6}$</td>
<td>$-4.60 \times 10^{-6}$</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

Table 1: Values attributed to gambles A, B, C, and D described in the main text, by entropic prospective satisficing measures (see Example 1), assuming different values for the target $\tau$. In bold are the preferred gambles in each pair for each target.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Gamble A</th>
<th>Gamble B</th>
<th>Gamble C</th>
<th>Gamble D</th>
</tr>
</thead>
<tbody>
<tr>
<td>55,000</td>
<td>$\infty$</td>
<td>3.76490</td>
<td>0.04798</td>
<td>0.00000</td>
</tr>
<tr>
<td>250,000</td>
<td>$\infty$</td>
<td>1.66770</td>
<td>0.00000</td>
<td>-0.46876</td>
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<tr>
<td>500,000</td>
<td>$\infty$</td>
<td>0.08759</td>
<td>-0.04440</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>695,000</td>
<td>$-\infty$</td>
<td>0.00000</td>
<td>-0.12513</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>2,000,000</td>
<td>$-\infty$</td>
<td>-1.19251</td>
<td>-1.36274</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

Table 2: Values attributed to gambles A, B, C, and D described in the main text, by homogenized entropic prospective satisficing measures (see Example 3), assuming different values for the target $\tau$. In bold are the preferred gambles in each pair for each target.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>Gamble A</th>
<th>Gamble B</th>
<th>Gamble C</th>
<th>Gamble D</th>
</tr>
</thead>
<tbody>
<tr>
<td>55,000</td>
<td>$\infty$</td>
<td>4.48864</td>
<td>0.08311</td>
<td>0.00000</td>
</tr>
<tr>
<td>250,000</td>
<td>$\infty$</td>
<td>3.91202</td>
<td>0.00000</td>
<td>-1.51413</td>
</tr>
<tr>
<td>500,000</td>
<td>$\infty$</td>
<td>0.10259</td>
<td>-0.69315</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>695,000</td>
<td>$-\infty$</td>
<td>0.00000</td>
<td>-1.02245</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>2,000,000</td>
<td>$-\infty$</td>
<td>-2.01490</td>
<td>-2.07944</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

Table 3: Values attributed to gambles A, B, C, and D described in the main text, by CVaR prospective satisficing measures (see Example 2), assuming different values for the target $\tau$. In bold are the preferred gambles in each pair for each target.
Table 4: Values attributed to gambles A, B, C and D given in the main text by entropic (columns 1-2), homogenous entropic (columns 3-4) and CVaR PSM (columns 5-6), as function of the target $\tau$. In bold are the preferred gambles in each pair for each target.
<table>
<thead>
<tr>
<th>Asset</th>
<th>$v_i$</th>
<th>$\tau_i$</th>
<th>$\nu_i$</th>
<th>$\nu_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>2</td>
<td>-30.0</td>
<td>6.0</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>-40.0</td>
<td>8.0</td>
<td>5.0</td>
<td>6.0</td>
</tr>
<tr>
<td>4</td>
<td>-50.0</td>
<td>10.0</td>
<td>8.0</td>
<td>9.0</td>
</tr>
<tr>
<td>5</td>
<td>-60.0</td>
<td>15.0</td>
<td>11.0</td>
<td>12.0</td>
</tr>
<tr>
<td>6</td>
<td>-100.0</td>
<td>20.0</td>
<td>15.0</td>
<td>16.0</td>
</tr>
</tbody>
</table>

Table 5: Supports $[v_i, \nu_i]$ of the distributions of assets’ percentage returns $V_i$ and the corresponding ranges $[\nu_i, \nu_i]$ for the expected returns for the portfolio choice example in Section 5.

<table>
<thead>
<tr>
<th>Asset</th>
<th>(\tau)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>(\rho)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>(\infty)</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>0.718</td>
<td>0.065</td>
<td>0.049</td>
<td>0.077</td>
<td>0.054</td>
<td>0.036</td>
<td>0.3220</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>0.435</td>
<td>0.130</td>
<td>0.099</td>
<td>0.155</td>
<td>0.109</td>
<td>0.073</td>
<td>0.1610</td>
<td></td>
</tr>
<tr>
<td>5.0</td>
<td>0.153</td>
<td>0.195</td>
<td>0.148</td>
<td>0.232</td>
<td>0.163</td>
<td>0.109</td>
<td>0.1073</td>
<td></td>
</tr>
<tr>
<td>6.0</td>
<td>0.000</td>
<td>0.192</td>
<td>0.164</td>
<td>0.292</td>
<td>0.210</td>
<td>0.142</td>
<td>0.0795</td>
<td></td>
</tr>
<tr>
<td>7.0</td>
<td>0.000</td>
<td>0.069</td>
<td>0.138</td>
<td>0.348</td>
<td>0.261</td>
<td>0.183</td>
<td>0.0585</td>
<td></td>
</tr>
<tr>
<td>9.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.350</td>
<td>0.361</td>
<td>0.289</td>
<td>0.0315</td>
<td></td>
</tr>
<tr>
<td>11.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.488</td>
<td>0.512</td>
<td>0.0146</td>
<td></td>
</tr>
<tr>
<td>14.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>0.0031</td>
<td></td>
</tr>
<tr>
<td>18.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.000</td>
<td>-0.0136</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Optimal asset allocation under the entropic PSM for various values of the target $\tau$ for the portfolio choice example in Section 5.