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Abstract

This paper analyzes a two-stage pairwise elimination contest with heterogeneous agents. It derives analytical expressions for equilibrium efforts in a setting where the two simultaneous stage-1 interactions are linked through endogenously determined continuation values.

Keywords

Multi-Stage Elimination Contest; Heterogeneity; Endogenous Continuation Value.

JEL Classification

C72, D72, J33.

1 Introduction

The multi-stage pairwise elimination (or knock-out) contest is well-known from sports – disciplines like baseball, boxing, hockey, soccer, tennis, or even chess make use of this structure in the so-called "playoff" stage. The structural feature of subsequent elimination is relevant in many other fields as well – just think of promotion tournaments on internal labor markets, or of multi-stage election campaigns like the one for US presidency, for example.

This paper analyzes a two-stage pairwise elimination contest with heterogeneous agents. When different types compete with each other in the two parallel stage-1 interactions, the value to win the first stage becomes endogenous – it depends on the identity of the stage-2 opponent, and therefore on the equilibrium behavior of contestants in the parallel stage-1 game. Thus, the two pairwise stage-1 interactions are non-trivially linked through endogenously determined continuation values, making it difficult to derive analytical expressions for equilibrium efforts. Using a lottery contest success function (CSF), we prove existence and uniqueness of pure strategy equilibrium in this game and derive analytical expressions for equilibrium efforts.

Rosen (1986) was the first to acknowledge that the equilibrium in a sequential elimination contest does not disassemble pairwise if contestants are heterogeneous; he suggests to use numerical simulations for the case where stage-1 interactions are heterogeneous to get insights on strategic incentives. Subsequent work on multi-stage contests with heterogeneous agents has circumvented the problem of endogenous continuation values in different ways: Stein and Rapoport (2004) restrict attention to an environment where stage-1 interactions are homogeneous, implying that the type-composition for the stage-2 interaction is predetermined so that the interaction effects across the parallel stage-1 games disappear; Harbaugh and Klumpp (2005) allow for interaction effects, but assume that total effort exerted by each participant over both stages of the contest is equal to some constant, reducing the dynamic optimization problem of contest participants to one of allocating the fixed effort endowment optimally across stages; Brown and Minor (2011) study an environment where stage-1 competitions are sequential rather than simultaneous, implying that backwards induction can be used to solve each interaction separately; and Groh, Moldovanu, Sela, and Sunde (2012) assume a perfectly discriminating CSF, implying that a positive runner-up prize for the stage-2 loser is required to get existence of equilibrium in stage 1. The present paper differs from Stein and Rapoport by studying an environment where stage-1 interactions are heterogeneous; from Harbaugh and Klumpp by analyzing a model where the effort in each stage is a costly choice variable and where the effort choice in stage 2 is unrestricted by the effort choice in stage 1; from Brown and Minor by investigating a setting in which the two stage-1 interactions take place simultaneously; and from Groh et al. by assuming an imperfectly discriminating CSF, implying that stage-1 winning probabilities, efforts, and payoffs are strictly positive for both types even without a runner-up prize.

2 The Model

Consider a two-stage elimination contest between four risk neutral agents; two agents have high effort cost $c_{\mathtt{H}}$, and the remaining two agents have low effort cost $c_{\mathtt{L}}$ ($0 < c_{\mathtt{L}} \leq c_{\mathtt{H}}$). Effort costs are common knowledge among contestants. All pairwise interactions – the two simultaneous semifinals in stage 1, and the final in stage 2 between stage-1 winners – are modeled as Tullock contests with a lottery CSF and linear effort costs. Thus, the winning probability p_i of agent i in an interaction with agent $j \neq i$ is given by the ratio of own effort x_i over total effort provided by i and j:

$$p_i(x_i, x_j) = \begin{cases} \frac{x_i}{x_i+x_j} & \text{if} \quad x_i + x_j > 0 \\ \frac{1}{2} & \text{if} \quad x_i + x_j = 0, \end{cases} \tag{1}$$

Agents compete for a total prize money of 1; the share $0 \leq \alpha \leq 0.5$ is awarded to the loser of the final, and the share $1 - \alpha$ to the winner.

Stage 2. Consider an interaction between agents i and j with costs of effort c_i and c_j. Agent i maximizes her stage-2 payoff Π_{i2} by choosing her effort level x_{i2}, taking the effort of agent j as given.[1] Formally, her maximization problem reads

$$\max_{x_{\mathtt{i}2} \geq 0} \Pi_{\mathtt{i}2}(x_{\mathtt{i}2}, x_{\mathtt{j}2}) = \frac{x_{\mathtt{i}2}}{x_{\mathtt{i}2} + x_{\mathtt{j}2}}[1 - 2\alpha] - c_{\mathtt{i}} x_{\mathtt{i}2} + \alpha.$$

The problem of agent j is symmetric but for the cost parameter c_j. The combination of first-order conditions, which are necessary and sufficient in any pairwise interaction for the lottery CSF (Cornes and Hartley 2005), delivers equilibrium efforts

$$x^*_{\mathtt{i}2} = \frac{c_{\mathtt{j}}}{(c_{\mathtt{i}} + c_{\mathtt{j}})^2}[1 - 2\alpha] \quad \text{and} \quad x^*_{\mathtt{j}2} = \frac{c_{\mathtt{i}}}{(c_{\mathtt{i}} + c_{\mathtt{j}})^2}[1 - 2\alpha]. \tag{2}$$

Inserting optimal actions in the two objective functions gives the expected equilibrium payoffs

$$\Pi_{\mathtt{i}2}(x^*_{\mathtt{i}2}, x^*_{\mathtt{j}2}) = \alpha + \frac{c_{\mathtt{j}}^2}{(c_{\mathtt{i}} + c_{\mathtt{j}})^2}[1 - 2\alpha] \quad \text{and} \quad \Pi_{\mathtt{j}2}(x^*_{\mathtt{j}2}, x^*_{\mathtt{i}2}) = \alpha + \frac{c_{\mathtt{i}}^2}{(c_{\mathtt{i}} + c_{\mathtt{j}})^2}[1 - 2\alpha]. \tag{3}$$

[1] The first subscript of Π and x indicates the player, the second one the stage.

Equations (2) and (3) characterize equilibrium efforts and payoffs, respectively, for any possible combination of cost types. In the sequel the type composition in stage 2 is put in parentheses. Specifically, we write $x^*_{im}(mn)$ for the stage-2 equilibrium effort of agent i in an environment where agent i has effort cost c_m while her opponent has effort cost c_n, with $(mn) \in \{(\mathtt{LL}), (\mathtt{LH}), (\mathtt{HL}), (\mathtt{HH})\}$. For future reference, we define $\Pi^*_{i2}(mn) \equiv \Pi_{\mathtt{i}2}(x^*_{\mathtt{i}2}(mn), x^*_{\mathtt{j}2}(mn))$ and $\Pi^*_{j2}(nm) \equiv \Pi_{\mathtt{j}2}(x^*_{\mathtt{j}2}(nm), x^*_{\mathtt{i}2}(nm))$.

Stage 1. Assume that effort costs of agents i and k are low, whereas j and l must bear high effort cost; the two stage-1 interactions are between agents i and j, and between k and l. Consider first the decision problems of low-cost agent i and high-cost agent j. Both search for their optimal stage-1 effort, taking the effort choice of the immediate opponent, the effort choices in the parallel stage-1 interaction, and equilibrium behavior in stage 2 as given. The optimization problems of i and j read

$$\max_{x_{i1}\geq 0} \Pi_{i1}(x_{i1}, x_{j1}|x_{k1}, x_{l1}) = \frac{x_{i1}}{x_{i1}+x_{j1}} \underbrace{\left[\frac{x_{k1}}{x_{k1}+x_{l1}}\Pi^*_{i2}(\mathtt{LL}) + \frac{x_{l1}}{x_{k1}+x_{l1}}\Pi^*_{i2}(\mathtt{LH})\right]}_{\equiv P_i(x_{k1},x_{l1})} - c_{\mathtt{L}} x_{i1}$$

$$\max_{x_{j1}\geq 0} \Pi_{j1}(x_{j1}, x_{i1}|x_{k1}, x_{l1}) = \frac{x_{j1}}{x_{i1}+x_{j1}} \underbrace{\left[\frac{x_{k1}}{x_{k1}+x_{l1}}\Pi^*_{j2}(\mathtt{HL}) + \frac{x_{l1}}{x_{k1}+x_{l1}}\Pi^*_{j2}(\mathtt{HH})\right]}_{\equiv P_j(x_{k1},x_{l1})} - c_{\mathtt{H}} x_{j1}\,.$$

The continuation values $P_i(x_{k1}, x_{l1})$ and $P_j(x_{k1}, x_{l1})$ of agents i and j, respectively, depend on the behavior of agents k and l in the parallel stage-1 interaction. Similarly, the continuation values $P_k(x_{i1}, x_{j1})$ and $P_l(x_{i1}, x_{j1})$ of agents k and l depend on the behavior of agents i and j. Thus, the two stage-1 interactions are non-trivially linked through endogenously determined continuation values. The reason is that – conditional on reaching stage 2 – an agent of either type has a higher expected payoff from meeting a high- rather than a low-cost opponent, since $\Pi^*_{i2}(\mathtt{LH}) \geq \Pi^*_{i2}(\mathtt{LL})$ and $\Pi^*_{j2}(\mathtt{HH}) \geq \Pi^*_{j2}(\mathtt{HL})$. However, each agent takes the probability that the opponent is a high-cost type as given, since it is determined in the parallel stage-1 interaction.

Combining the first-order conditions of direct opponents delivers two expressions which define a relation between equilibrium effort choices *within* each interaction, namely

$$\frac{x_{i1}}{x_{j1}} = \underbrace{\frac{c_{\mathtt{H}}}{c_{\mathtt{L}}}}_{\text{cost het.}} * \underbrace{\frac{P_i(x_{k1}, x_{l1})}{P_j(x_{k1}, x_{l1})}}_{\text{valuation heterogeneity}} \quad \text{and} \quad \frac{x_{k1}}{x_{l1}} = \underbrace{\frac{c_{\mathtt{H}}}{c_{\mathtt{L}}}}_{\text{cost het.}} * \underbrace{\frac{P_k(x_{i1}, x_{j1})}{P_l(x_{i1}, x_{j1})}}_{\text{valuation heterogeneity}}\,. \tag{4}$$

Equation-system (4) shows that the stage-1 equilibrium effort of a high-cost agent is lower than the stage-1 effort of a low-cost contestant for two reasons: First, high-cost agents invest less than their low-cost opponents due to their exogenous strategic disadvantage in the current interaction; this "cost heterogeneity" effect is present in any interaction between heterogeneous contestants. Second, agents with high costs provide less effort in stage-1 than their low-cost opponents due to "valuation heterogeneity", which emerges because the continuation value of a high-cost agent is lower than the continuation value of a low-cost agent. Thus, the cost heterogeneity effect is augmented by a dynamic valuation heterogeneity effect in stage 1. Inserting the continuation values in (4) and simplifying gives

$$\frac{x_{i1}}{x_{j1}} = \frac{c_{\mathtt{H}}}{c_{\mathtt{L}}} * \frac{\Pi^*_{\mathtt{L}2}(\mathtt{LL}) * \frac{x_{k1}}{x_{l1}} + \Pi^*_{\mathtt{L}2}(\mathtt{LH})}{\Pi^*_{\mathtt{H}2}(\mathtt{HL}) * \frac{x_{k1}}{x_{l1}} + \Pi^*_{\mathtt{H}2}(\mathtt{HH})} \quad \text{and} \quad \frac{x_{k1}}{x_{l1}} = \frac{c_{\mathtt{H}}}{c_{\mathtt{L}}} * \frac{\Pi^*_{\mathtt{L}2}(\mathtt{LL}) * \frac{x_{i1}}{x_{j1}} + \Pi^*_{\mathtt{L}2}(\mathtt{LH})}{\Pi^*_{\mathtt{H}2}(\mathtt{HL}) * \frac{x_{i1}}{x_{j1}} + \Pi^*_{\mathtt{H}2}(\mathtt{HH})}\,, \tag{5}$$

where we use the definition $\Pi^*_{\mathtt{L}2}(\mathtt{LH}) \equiv \Pi^*_{i2}(\mathtt{LH}) = \Pi^*_{k2}(\mathtt{LH})$, and similarly for $\Pi^*_{\mathtt{L}2}(\mathtt{LL})$, $\Pi^*_{\mathtt{H}2}(\mathtt{HL})$, and $\Pi^*_{\mathtt{H}2}(\mathtt{HH})$. System (5) consists of two equations in the two unknowns $\frac{x^*_{i1}}{x^*_{j1}}$ and $\frac{x^*_{k1}}{x^*_{l1}}$. The equations are symmetric, since agents of the same cost-type face identical optimization problems *across* the two stage-1 interactions. Thus, $x^*_{\mathtt{L}1} \equiv x^*_{i1} = x^*_{k1}$ and $x^*_{\mathtt{H}1} \equiv x^*_{j1} = x^*_{l1}$ in a symmetric equilibrium.

Proposition 1 (Existence and Uniqueness)**.** *A symmetric subgame perfect Nash-equilibrium in pure strategies exists for any degree of heterogeneity* $0 < c_L \leq c_H$*, and it is unique.*

Proof. Any contest with two heterogeneous participants has a unique, interior equilibrium for the chosen CSF (Cornes and Hartley 2005). Consequently, each of the two pairwise stage-1 interactions has a unique equilibrium for each pair of continuation values. It remains to be shown is that the two expressions in (5) can be satisfied *jointly* such that effort choices are mutually optimal across the two stage-1 interactions. Thus, we must show that the graphs of the two relations in (5) have a unique intersection in the domain defined by $\frac{x^*_{j1}}{x^*_{i1}} \in [0,1]$ and $\frac{x^*_{l1}}{x^*_{k1}} \in [0,1]$. It suffices to consider this domain, since effort cost and continuation value differences imply $x^*_{i1} \geq x^*_{j1}$ and $x^*_{k1} \geq x^*_{l1}$. Rearranging (5), we obtain

$$\frac{x_{j1}}{x_{i1}} = \underbrace{\frac{c_{\mathtt{L}}}{c_{\mathtt{H}}} * \frac{\Pi^*_{\mathtt{H}2}(\mathtt{HL}) + \Pi^*_{\mathtt{H}2}(\mathtt{HH}) * \frac{x_{l1}}{x_{k1}}}{\Pi^*_{\mathtt{L}2}(\mathtt{LL}) + \Pi^*_{\mathtt{L}2}(\mathtt{LH}) * \frac{x_{l1}}{x_{k1}}}}_{\equiv R\left(\frac{x_{l1}}{x_{k1}}\right)} \quad \text{and} \quad \frac{x_{l1}}{x_{k1}} = \underbrace{\frac{c_{\mathtt{L}}}{c_{\mathtt{H}}} * \frac{\Pi^*_{\mathtt{H}2}(\mathtt{HL}) + \Pi^*_{\mathtt{H}2}(\mathtt{HH}) * \frac{x_{j1}}{x_{i1}}}{\Pi^*_{\mathtt{L}2}(\mathtt{LL}) + \Pi^*_{\mathtt{L}2}(\mathtt{LH}) * \frac{x_{l1}}{x_{k1}}}}_{\equiv Z\left(\frac{x_{j1}}{x_{i1}}\right)}\,.$$

Taking the first derivatives of the functions $R(\cdot)$ and $Z(\cdot)$ with respect to the only argument reveals that both functions have a strictly positive slope for $c_{\mathtt{H}} > c_{\mathtt{L}}$, while the slope is zero for $c_{\mathtt{H}} = c_{\mathtt{L}}$. Thus, both functions are strictly monotonic in their respective argument for $c_{\mathtt{H}} > c_{\mathtt{L}}$, and inverse functions do exist. Consider the inverse function

$$R^{-1}\left(\frac{x_{j1}}{x_{i1}}\right) = \frac{x_{l1}}{x_{k1}} = \frac{c_{\mathtt{H}}\Pi^*_{\mathtt{L}2}(\mathtt{LL}) * \frac{x_{j1}}{x_{i1}} - c_{\mathtt{L}}\Pi^*_{\mathtt{H}2}(\mathtt{HL})}{c_{\mathtt{L}}\Pi^*_{\mathtt{H}2}(\mathtt{HH}) - c_{\mathtt{H}}\Pi^*_{\mathtt{L}2}(\mathtt{LH}) * \frac{x_{j1}}{x_{i1}}}.$$

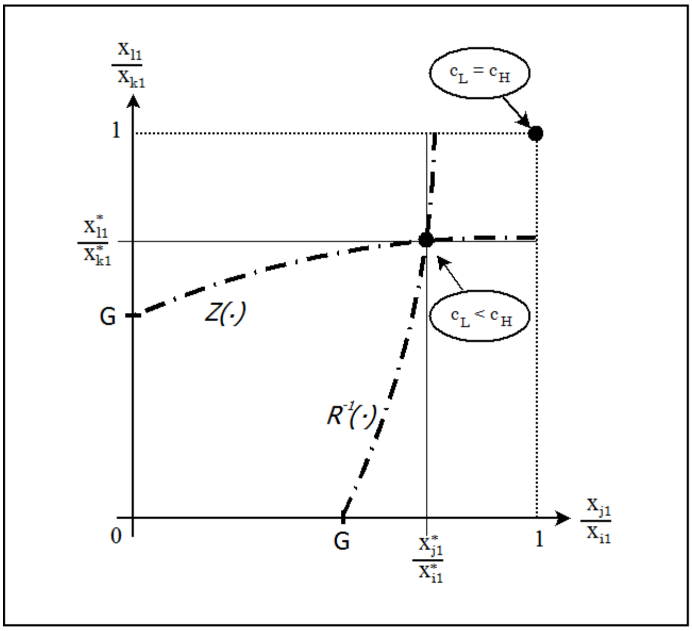

Figure 1: Interdependence across Stage-1 Interactions

By definition of the inverse function, $R^{-1}(\cdot)$ must also be strictly monotonic. Furthermore, $R^{-1}(\cdot) = 0$ for $\frac{x_{j1}}{x_{i1}} = \frac{c_{\mathtt{L}}\Pi^*_{\mathtt{H}2}(\mathtt{HL})}{c_{\mathtt{H}}\Pi^*_{\mathtt{L}2}(\mathtt{LL})} \equiv G$ and $R^{-1}(\cdot) = 1$ for $\frac{x_{j1}}{x_{i1}} = \frac{c_{\mathtt{L}}\Pi^*_{\mathtt{H}2}(\mathtt{HH})}{c_{\mathtt{H}}\Pi^*_{\mathtt{L}2}(\mathtt{LH})} \equiv W$, where $0 < G < W < 1$. Moreover, $Z(0) = G$ and $Z(1) = W$ due to symmetry. Thus, since $R^{-1}(\cdot)$ and $Z(\cdot)$ are both continuous and strictly increasing, their graphs must intersect exactly once in the domain $\frac{x_{j1}}{x_{i1}} \in [0, 1]$ by the intermediate value theorem of calculus. $\square$

Figure 1 sketches the main argument of the proof and illustrates the effect of heterogeneity between contestants in stage 1 graphically: In homogeneous settings ($c_{\mathtt{L}} = c_{\mathtt{H}}$), the ratios of equilibrium efforts in the two stage-1 interactions are *independent* from one another. Thus, the equilibrium is a fixed point of two straight lines. In contrast, the ratios of equilibrium efforts in the two stage-1 interactions are *interdependent* in heterogeneous settings ($c_{\mathtt{L}} < c_{\mathtt{H}}$). In this case, the equilibrium is the fixed point of two reaction functions with positive slopes.

Imposing the symmetry condition $\frac{x_{i1}}{x_{j1}} = \frac{x_{k1}}{x_{l1}} \equiv \frac{x_{\mathtt{L}1}}{x_{\mathtt{H}1}}$ on equation-system (5) gives a quadratic equation in $\frac{x_{\mathtt{L}1}}{x_{\mathtt{H}1}}$. Inserting formal expressions for expected stage-2 equilibrium payoffs from (3) and solving for the equilibrium ratio of efforts delivers

$$\frac{x^*_{\mathtt{L}1}}{x^*_{\mathtt{H}1}} = \underbrace{\frac{c_{\mathtt{H}}}{c_{\mathtt{L}}} * \frac{(c_{\mathtt{H}} - c_{\mathtt{L}})(c_{\mathtt{L}} + c_{\mathtt{H}})^2(1 - 2\alpha) + \sqrt{\phi(\alpha, c_{\mathtt{L}}, c_{\mathtt{H}})}}{8c_{\mathtt{H}}[(1 - \alpha)c_{\mathtt{L}}^2 + \alpha c_{\mathtt{H}} c_{\mathtt{L}} + \alpha c_{\mathtt{H}}^2]}}_{\equiv F^*(\alpha, c_{\mathtt{L}}, c_{\mathtt{H}})}. \tag{6}$$

where $\phi(\alpha, c_{\mathtt{L}}, c_{\mathtt{H}}) = 64 c_{\mathtt{H}} c_{\mathtt{L}}[\alpha c_{\mathtt{H}}^2 + \alpha c_{\mathtt{H}} c_{\mathtt{L}} + (1-\alpha)c_{\mathtt{L}}^2][\alpha c_{\mathtt{L}}^2 + \alpha c_{\mathtt{H}} c_{\mathtt{L}} + (1-\alpha)c_{\mathtt{H}}^2] + (c_{\mathtt{L}} - c_{\mathtt{H}})^2(c_{\mathtt{L}} + c_{\mathtt{H}})^4(1-2\alpha)$. $F^*(\alpha, c_{\mathtt{L}}, c_{\mathtt{H}})$ defines the unique ratio of (positive) equilibrium efforts which ensures that effort choices are mutually optimal *within* and *across* the two stage-1 interactions. Inserting $\frac{x^*_{\mathtt{L}1}}{x^*_{\mathtt{H}1}}$ from (6) in the first-order optimality conditions gives stage-1 equilibrium efforts

$$x^*_{\mathtt{L}1} \equiv x^*_{i1} = x^*_{k1} = \frac{(c_\mathtt{L}+c_\mathtt{H})^2 F^*(\alpha, c_\mathtt{H}, c_\mathtt{L})^2 + 4c_\mathtt{H}^2 F^*(\alpha, c_\mathtt{H}, c_\mathtt{L})}{4c_\mathtt{L}(c_\mathtt{L}+c_\mathtt{H})^2[1+F^*(\alpha, c_\mathtt{L}, c_\mathtt{H})]^3}, \tag{7}$$

$$x^*_{\mathtt{H}1} \equiv x^*_{j1} = x^*_{l1} = \frac{(c_\mathtt{L}+c_\mathtt{H})^2 F^*(\alpha, c_\mathtt{H}, c_\mathtt{L}) + 4c_\mathtt{L}^2 F^*(\alpha, c_\mathtt{H}, c_\mathtt{L})^2}{4c_\mathtt{H}(c_\mathtt{L}+c_\mathtt{H})^2[1+F^*(\alpha, c_\mathtt{L}, c_\mathtt{H})]^3}. \tag{8}$$

3 Conclusion

This paper has derived analytical expressions for equilibrium efforts in a two-stage elimination contest where pairwise stage-1 interactions are non-trivially linked though endogenously determined continuation values. The solution presented in this paper could be used to investigate the impact of simplifying assumptions made in the existing literature, or to compare the effect of heterogeneity in different contest formats – analytical solution for static pooling contests with heterogeneous agents are already available, for example. These research questions, as well as generalization of the model with respect to the number of stages, types, or stage-1 interactions, are left for future work.

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