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Abstract

The paper presents GARCH option pricing models with Meixner-distributed innovations. The risk-neutral dynamics are derived by means of the conditional Esscher transform. Assessing the option pricing performance both in-sample and out-of-sample, we find that the models compare favorably against the benchmark models. Simulations suggest that the driver of these results is the impact of conditional skewness and conditional excess kurtosis on option prices.

Keywords

GARCH models, Meixner distribution, Esscher transform, option pricing.

JEL Classification

G13, C22.

1 Introduction

Continuous sample paths of asset prices and normality of returns have played a central role in the pricing theory of financial derivatives. Samuelson (1965) introduced the geometric Brownian motion, which is fundamental to the seminal papers of Black and Scholes (1973) and Merton (1973). As is widely documented, however, the geometric Brownian motion is not able to cope with volatility clustering and the stochastic nature of the volatility. To incorporate the stochasticity of volatility, three streams of literature have emerged: continuous-time stochastic volatility models based on diffusion processes, such as that of Hull and White (1987); discrete-time stochastic volatility models as originated by Taylor (2008); and discrete-time generalized autoregressive conditionally heteroskedastic (GARCH) models as introduced by Engle (1982) and Bollerslev (1986). As a distinct advantage of GARCH models, the specification of volatility dynamics as a function of past returns allows one to filter the volatility process very easily. This makes GARCH models particularly attractive from the perspective of estimation and simulation.

Empirical evidence of equity markets shows that standardized GARCH residuals still exhibit negative skewness and excess kurtosis, even when asymmetric variance dynamics are employed, such as in the threshold GARCH model, which is able to reproduce unconditional skewness and excess kurtosis. Conditional negative skewness and excess kurtosis therefore needs to be introduced by means of skewed and heavy-tailed distributed innovations. Moreover, these distributional features are known to be crucial for accurately fitting option price data. Accordingly, a number of alternative innovations distributions have been suggested for GARCH-based option pricing models, such as the shifted Gamma in Siu et al. (2004), the Inverse Gaussian in Christoffersen et al. (2006), the Generalized Hyperbolic in Chorro et al. (2012), and the Normal Inverse Gaussian (NIG) in Stentoft (2008) and Badescu et al. (2011).

In this work, we develop a GARCH option pricing model with innovations that are drawn from the Meixner distribution. The Meixner distribution is skewed and heavy-tailed, belongs to the class of infinitely divisible distributions and therefore gives rise to a Lévy process called the Meixner process that is studied by Schoutens and Teugels (1998) and Grigelionis (1999). Schoutens (2002) demonstrates that the Meixner process fits historical return data of equity indices very well and that it exhibits promising properties for option pricing. Motivated by these findings, we incorporate the Meixner distribution into a GARCH framework for option pricing.

We proceed by specifying a GARCH-in-mean model for the underlying asset's return process under the historical measure using alternative variance dynamics, such as the GARCH and the threshold GARCH (TGARCH) of Glosten et al. (1993). The GARCH-in-mean specification is meant to capture time-varying variance risk premia – see Engle et al. (1987). We then employ the conditional Esscher transform of Bühlmann et al. (1996) to derive the risk-neutral pricing measure. More precisely, because the moment-generating function of the Meixner distribution exists and is analytically tractable, we derive an analytical expression of the Esscher transform and the Radon-Nikodym derivatives process and can characterize the risk-neutral dynamics of logarithmic returns of the underlying asset, which turn out to be conditional Meixner with time-varying parameters. As in Fengler et al. (2012), these results allow us to estimate the model from time series data and to price options by simulating the transition density of stock prices and the Radon-Nikodym process under the historical measure jointly. Consequently, we do not require a calibration of model parameters under the risk-neutral probability measure.

Despite its evident suitability, to date, the Meixner distribution has not been much used for asset pricing. Grigoletto and Provasi (2008) suggest GARCH models with Meixner innovations to describe financial return data. Moolman (2008) considers a Meixner GARCH model for option valuation and relies on the local risk-neutral valuation relationship (LRNVR) of Duan (1995) to derive the risk-neutral pricing measure. The LRNVR, however, is only applicable if innovations are conditionally normal. In this work, we therefore develop a rigorous approach that is based on the Esscher transform. The use of the Esscher transform to characterize the pricing measure is a well-established technique for non-normal innovation processes – see, e.g., Gerber and Shiu (1994), Siu et al. (2004), Mercuri (2008), Badescu and Kulperger (2008), Christoffersen et al. (2010), Badescu et al. (2011) and Chorro et al. (2012). The martingale measure obtained by the conditional Esscher transform corresponds to a specific exponential affine stochastic discount factor (Gouriéroux and Monfort, 2007).

In our empirical applications, we assess the goodness of fit and the option pricing performance of the GARCH models in-sample and out-of-sample using S&P500 index data and price data of options written on the S&P500 index. We find that the model compares favorably against most of its competitors and is at par with the NIG-TGARCH model studied in Badescu et al. (2011). In order to better understand the performance of the Meixner models, we study the patterns of Black-Scholes implied volatility of a TGARCH model with normal and Meixner innovations. We find that the implied volatility patterns are driven by the shape parameters of the conditional Meixner distribution. The simulations suggest that aside from specific forms of heteroskedasticity, conditional skewness and conditional excess kurtosis are decisive for accurate option valuation.

The paper is organized as follows. Section 2 presents the main properties of the Meixner distribution. In Section 3, we introduce GARCH-based option pricing models with Meixner innovations and characterize the pricing measure. The empirical part and supportive simulations are provided in Sections 4 and 5. Section 6 concludes. An appendix details the Meixner random variables generator.

2 The Meixner distribution

A random variable X has the Meixner distribution $MD(a, b, m, d)$ if its probability density function is given by:

$$f_{MD}(x; a, b, m, d) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)} \exp\left(b\,\frac{x-m}{a}\right) \left|\Gamma\left(d + \mathrm{i}\,\frac{x-m}{a}\right)\right|^{2}, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function, $\mathrm{i} = \sqrt{-1}$, $a > 0$, $-\pi < b < \pi$, $m \in \mathbb{R}$, $d > 0$ and $x \in \mathbb{R}$ – see Schoutens (2002). The Meixner distribution is infinitely divisible and gives rise to a Lévy process studied in Schoutens and Teugels (1998) and Grigelionis (1999).

In (2.1), the parameter d influences the peakedness and the parameter b affects the skewness of the Meixner distribution. Parameters a and m define scale and location, respectively. Moments of all orders exist. In particular,

$$\begin{aligned}
\mathrm{E}[X] &= m + ad\tan\left(\frac{b}{2}\right) = \mu_{MD}\,, \\
\mathrm{Var}[X] &= \frac{a^2 d}{2}\,\frac{1}{\cos^2(b/2)} = \sigma^2_{MD}\,, \\
\mathrm{Skew}[X] &= \sqrt{\frac{2}{d}}\,\sin\left(\frac{b}{2}\right), \\
\mathrm{Kurt}[X] &= 3 + \frac{3 - 2\cos^2(b/2)}{d}\,.
\end{aligned} \tag{2.2}$$

The distribution is symmetric for $b = 0$, skewed to the left (right) for $b < 0$ ($b > 0$). The kurtosis of the Meixner distribution always exceeds the kurtosis of the normal distribution.

The Meixner distribution has semi-heavy tails. More specifically, one has the following tail behavior (Grigelionis, 2001):

$$\begin{aligned}
f_{MD}(x; a, b, m, d) &\sim C_-|x|^\rho \exp(-\sigma_-|x|) \quad \text{as} \quad x \to -\infty\,, \\
f_{MD}(x; a, b, m, d) &\sim C_+|x|^\rho \exp(-\sigma_+|x|) \quad \text{as} \quad x \to +\infty\,,
\end{aligned}$$

for some $\rho \in \mathbb{R}$ and C_-, C_+, σ_-, $\sigma_+ \geqslant 0$, where $\rho = 2d - 1\,,\quad \sigma_- = (\pi - b)/a\,,\quad \sigma_+ = (\pi + b)/a$.

The MGF of the Meixner distribution $MD(a, b, m, d)$ exists and can be derived from the characteristic function (Grigelionis, 2001). It is given by

$$\mathrm{E}[e^{uX}] = e^{mu} \left(\frac{\cos\left(\frac{b}{2}\right)}{\cos\left(\frac{b+au}{2}\right)} \right)^{2d}, \quad u \in \left(\frac{-\pi - b}{a}, \frac{\pi - b}{a} \right). \tag{2.3}$$

Suppose that $X \sim MD(a, b, m, d)$ and $Y = AX + B$ with $A \in \mathbb{R}_{>0}$ and $B \in \mathbb{R}$. Because Y has the density $f_Y(x) = \frac{1}{A} f_X\left(\frac{x-B}{A}\right)$, we observe that

$$Y \sim MD(Aa, b, Am + B, d) \,. \tag{2.4}$$

Hence, the parameters b and d are invariant and the Meixner distribution is closed under affine transformations. This property allows one to define a zero mean, unit variance Meixner distribution as

$$\frac{X - \mu_{MD}}{\sigma_{MD}} \sim MD\left(\frac{a}{\sigma_{MD}}, \, b \, , \frac{m - \mu_{MD}}{\sigma_{MD}}, \, d \right) . \tag{2.5}$$

As can be seen from (2.2), the parameters a and m cancel out. Hence, they can be expressed as functions of b and d:

$$a = \sqrt{\frac{2\cos^2\left(\frac{b}{2}\right)}{d}}; \tag{2.6}$$

$$m = -\, ad \tan\left(\frac{b}{2}\right) . \tag{2.7}$$

In Figure 1, we contrast the standard normal with the Meixner density function with zero mean and unit variance, which exhibits non-zero skewness and excess kurtosis.

Because the moments take so simple forms, the Meixner parameters can be estimated easily by method of moments techniques. In particular, in our applications, we estimate b and d by equating the theoretical skewness and kurtosis with their sample analogues. The parameters a and m are obtained from (2.6) and (2.7). These moment-based estimators have proved to be useful as initial values for the maximum likelihood estimation of our models with Meixner-distributed innovations (see Section 4).

3 GARCH-based option pricing models with Meixner innovations

3.1 GARCH specifications

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}$ be the information structure, where $\mathcal{F}_t$ represents the information set of all market information at time $t = 1, \ldots, T$. Under the historical measure $\mathbb{P}$, consider a market with two assets. The riskless bond price process is specified by

$$B_t = B_{t-1}\, e^{r_t}, \tag{3.1}$$

where $\{r_t\}$ is a predictable process which describes the daily risk-free rate. We assume $r_t = r$ to be constant for simplicity. The risky stock price process is given by

$$S_t = S_{t-1}\, e^{X_t}, \tag{3.2}$$

where $\{X_t\}$, the logarithmic return of the stock, is adapted to $\{\mathcal{F}_t\}$. We assume that X_t and the conditional variance h_t have the following dynamics under the historical measure $\mathbb{P}$:

$$\begin{cases} X_t &= \mu_t + \varepsilon_t\,, \\ h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \gamma_1 \mathbb{1}(\varepsilon_{t-1} < 0)\varepsilon_{t-1}^2 + \beta_1 h_{t-1}\,, \end{cases} \tag{3.3}$$

where μ_t is a $\mathcal{F}_{t-1}$-measurable conditional mean, $\varepsilon_t = \sqrt{h_t}\,\frac{Y_t - \mu_{MD}}{\sigma_{MD}}$ is an innovation process with zero mean and conditional variance h_t, and $Y_t \overset{iid.}{\sim} MD(a, b, m, d)$. For $\gamma_1 = 0$, the conditional variance follows a standard GARCH specification, and for $\gamma_1 \neq 0$ we have the TGARCH model of Glosten et al. (1993).

Grigoletto and Provasi (2008) consider a general GARCH-type model, namely an AR-APARCH model, which is driven by Meixner innovations. Unlike them, we consider a GARCH-in-mean model, in which the conditional variance feeds into the mean equation.

This establishes a time-varying risk premium as a function of the conditional variance (Engle et al., 1987). We set $\mu_t = r + \lambda\sqrt{h_t}$, where λ is a parameter. For normal and shifted Gamma innovations, we consider $\mu_t = r + \lambda\sqrt{h_t} - 0.5h_t$ as in Duan (1995).

Because the Meixner distribution is closed under affine transformations, the conditional distribution of returns is given by:

$$X_t|\mathcal{F}_{t-1} \sim MD\left(\frac{a\sqrt{h_t}}{\sigma_{MD}}, b, \frac{(m-\mu_{MD})\sqrt{h_t}}{\sigma_{MD}} + \mu_t, d\right). \tag{3.4}$$

As discussed in Section 2, the parameters a and m are redundant, but are functions of b and d via (2.6) and (2.7). We keep them here for the sake of clearness.

Given (2.3), it follows that the conditional MGF of X_t is

$$\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(z) = \mathrm{E}^{\mathbb{P}}[e^{zX_t}|\mathcal{F}_{t-1}] = \exp\left\{z\left(\mu_t + \frac{m-\mu_{MD}}{\sigma_{MD}}\sqrt{h_t}\right)\right\}\left(\frac{\cos\left(\frac{b}{2}\right)}{\cos\left(\frac{b+\frac{a\sqrt{h_t}}{\sigma_{MD}}z}{2}\right)}\right)^{2d}. \tag{3.5}$$

The conditional MGF is in closed form and plays a crucial role in the derivation of the equivalent martingale measure by means of the conditional Esscher transform.

3.2 The conditional Esscher transform

Gerber and Shiu (1994) introduce the change of measure by means of the Esscher transform. The conditional Esscher transform is first used in Bühlmann et al. (1996) and Siu et al. (2004). An advantage of the change of measure by means of the conditional Esscher transform over Duan's LRNVR is that it is applicable to any distribution whenever its MGF exists.

Assume, for all $t = 1, \ldots, T$, $\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(z) = \mathrm{E}^{\mathbb{P}}[e^{zX_t}|\mathcal{F}_{t-1}] < \infty$. As in Siu et al. (2004) and in Christoffersen et al. (2010), we define a stochastic process $\{\mathcal{L}_t\}$, $t = 1, \ldots, T$, by

$$\mathcal{L}_t = \prod_{k=1}^{t} \frac{e^{\theta_k X_k}}{\mathcal{M}_{X_k|\mathcal{F}_{k-1}}(\theta_k)}, \tag{3.6}$$

where $\{\theta_k\}$ is a predictable process, $\mathcal{L}_0 = 1$ and $\mathrm{E}^{\mathbb{P}}[\,\mathcal{L}_t\,] = 1$. Evidently, $\mathrm{E}^{\mathbb{P}}[\,\mathcal{L}_t \,|\, \mathcal{F}_{t-1}] = \mathcal{L}_{t-1}$, i.e., $\{\mathcal{L}_t\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. By these properties, $\{\mathcal{L}_t\}$ defines a change of measure by means of the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{L}_T = \prod_{k=1}^{T} \frac{e^{\theta_k X_k}}{\mathcal{M}_{X_k|\mathcal{F}_{k-1}}(\theta_k)}\,, \tag{3.7}$$

where $\mathbb{Q}$ is the risk-neutral or equivalent martingale measure. Taking into account that under $\mathbb{Q}$ the drift of the asset price process is equal to r, we obtain the relation

$$\mathrm{E}^{\mathbb{P}}\left[\frac{e^{\theta_t X_t}}{\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(\theta_t)} e^{X_t} \,\middle|\, \mathcal{F}_{t-1}\right] = \mathrm{E}^{\mathbb{Q}}\left[e^{X_t} \,\middle|\, \mathcal{F}_{t-1}\right] = e^{r}\,. \tag{3.8}$$

Hence, the predictable process $\{\theta_t^*\}$ which solves the martingale Esscher equation

$$\frac{\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(\theta_t + 1)}{\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(\theta_t)} = e^{r} \tag{3.9}$$

parametrizes the corresponding risk-neutral measure. The existence of the solution is guaranteed by the existence of $\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(z)$, see Grigelionis (1999). Christoffersen et al. (2010) prove uniqueness if the conditional log MGF is strictly convex and twice differentiable. From (3.8), it follows that the conditional MGF of log-returns under $\mathbb{Q}$ is given by

$$\mathcal{M}^{\mathbb{Q}}_{X_t|\mathcal{F}_{t-1}}(z) = \frac{\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(\theta_t^* + z)}{\mathcal{M}_{X_t|\mathcal{F}_{t-1}}(\theta_t^*)}\,. \tag{3.10}$$

Besides the explicit form of the conditional distribution of log-returns, this relation allows us to characterize the risk-neutral dynamics of returns in Section 3.3.

3.3 The risk-neutral dynamics

Here we derive the risk-neutral dynamics under the GARCH framework with the Meixner innovations. To find the Esscher parameter θ_t^*, insert (3.5) into (3.9). This yields:

$$\exp\left(\mu_t + \frac{m - \mu_{MD}}{\sigma_{MD}}\sqrt{h_t}\right)\left(\frac{\cos\left(\frac{b + \frac{a\sqrt{h_t}}{\sigma_{MD}}\theta_t^*}{2}\right)}{\cos\left(\frac{b + \frac{a\sqrt{h_t}}{\sigma_{MD}}(\theta_t^* + 1)}{2}\right)}\right)^{2d} = e^{r}\,. \tag{3.11}$$

Straightforward calculations show

$$\cos\left(\frac{\delta_t(\theta_t^* + 1) + b}{2}\right) = \zeta_t \cos\left(\frac{\delta_t\,\theta_t^* + b}{2}\right), \tag{3.12}$$

where $\delta_t = \frac{a\sqrt{h_t}}{\sigma_{MD}}$ and $\zeta_t = \exp\left\{\left(\mu_t - r + \frac{m-\mu_{MD}}{\sigma_{MD}}\sqrt{h_t}\right)/(2d)\right\}$. Because $\mu_t = r + \lambda\sqrt{h_t}$, the risk-free rate r cancels out. Using a result in Schoutens (2002, p. 16), we find

$$\theta_t^* = -\frac{2}{\delta_t}\arctan\left(\frac{\zeta_t - \cos\left(\frac{\delta_t}{2}\right)}{\sin\left(\frac{\delta_t}{2}\right)}\right) - \frac{b}{\delta_t}. \tag{3.13}$$

Based on (3.10), under $\mathbb{Q}$, the conditional MGF of log-returns can be written as

$$\mathcal{M}^{\mathbb{Q}}_{X_t|\mathcal{F}_{t-1}}(z) = \exp\left\{z\left(\mu_t + \frac{m-\mu_{MD}}{\sigma_{MD}}\sqrt{h_t}\right)\right\}\left(\frac{\cos\left(\frac{b+\frac{a\sqrt{h_t}}{\sigma_{MD}}\theta_t^*}{2}\right)}{\cos\left(\frac{b+\frac{a\sqrt{h_t}}{\sigma_{MD}}(\theta_t^*+z)}{2}\right)}\right)^{2d}. \tag{3.14}$$

Hence, under $\mathbb{Q}$, the conditional MGF specifies a return distribution of the form

$$X_t|\mathcal{F}_{t-1} \sim MD\left(\frac{a\sqrt{h_t}}{\sigma_{MD}}, b_t^*, \frac{(m-\mu_{MD})\sqrt{h_t}}{\sigma_{MD}} + \mu_t, d\right), \tag{3.15}$$

where the time-varying shape parameter b_t^* is defined by

$$b_t^* = b + \frac{a\sqrt{h_t}}{\sigma_{MD}}\theta_t^*. \tag{3.16}$$

Importantly, the conditional Esscher transform shifts only the shape parameter b of the Meixner distribution under $\mathbb{P}$ to a conditional variance-dependent parameter b_t^* under $\mathbb{Q}$, while keeping all other parameters constant.

Under $\mathbb{Q}$, the mean equation of the logarithmic returns can be written as:

$$X_t = \mu_t + \varepsilon_t, \tag{3.17}$$

where ε_t has the conditional Meixner distribution

$$\varepsilon_t|\mathcal{F}_{t-1} \sim MD\left(\frac{a\sqrt{h_t}}{\sigma_{MD}}, b_t^*, \frac{(m-\mu_{MD})\sqrt{h_t}}{\sigma_{MD}}, d\right). \tag{3.18}$$

Because the logarithmic returns preserve the Meixner distribution under the risk-neutral measure, the next equations can be exploited to simulate the price process of the underlying asset price under $\mathbb{Q}$ by means of Monte Carlo simulations:

$$X_t = \mu_t + \varepsilon_t^* \,, \tag{3.19}$$

$$h_t = \alpha_0 + \alpha_1 (\varepsilon_{t-1}^*)^2 + \beta_1 h_{t-1} \,, \tag{3.20}$$

where $\varepsilon_t^* = \sqrt{h_t} \frac{Y_t^* - \mu_{MD}}{\sigma_{MD}}$ and $Y_t^* \sim MD(a, b_t^*, m, d)$. The Meixner variables Y_t^* are neither independent nor identically distributed.

It should be noted that under $\mathbb{Q}$ the conditional expectation

$$\mathrm{E}^{\mathbb{Q}}[X_t | \mathcal{F}_{t-1}] = \mu_t + \frac{\sin\left(\frac{b_t^* - b}{2}\right)}{\sin\left(\frac{b_t^*}{2}\right)} \sqrt{2 d h_t} \tag{3.21}$$

and the conditional variance of the log-returns

$$\mathrm{Var}^{\mathbb{Q}}[X_t | \mathcal{F}_{t-1}] = \frac{\cos^2\left(\frac{b}{2}\right)}{\cos^2\left(\frac{b_t^*}{2}\right)} h_t = \frac{1 + \cos(b)}{1 + \cos(b_t^*)} h_t \tag{3.22}$$

become time-varying, which leads to highly non-linear risk-neutral dynamics of the conditional variance process. Because h_t is not the conditional variance process under $\mathbb{Q}$, (3.20) is only an updating equation and no longer has the interpretation of a conditional variance process. See Chorro et al. (2012, p. 1087) for similar observations on their generalized hyperbolic GARCH model.

Referring to Equations (2.2), we can conclude that the time-varying parameter b_t^* implies a time-varying skewness and kurtosis under the risk-neutral measure. Moreover, because stock returns are typically negatively skewed, i.e., $b < 0$, the Esscher parameter (3.13) yields negative values for $-\pi/2 < b < 0$. Hence, from (3.16), in most empirical cases, we obtain $b_t^* < b < 0$. This result implies that the risk-neutral density is more negatively skewed and has a larger conditional variance than does the historical one. These observations give important insights into the dynamics of the risk-neutralized process and how it is distinct from GARCH models with normal innovations.

3.4 Interpretation of the Esscher transform as an exponential affine stochastic discount factor

Gouriéroux and Monfort (2007) consider asset pricing using a stochastic discount factor (SDF). If an agent makes investments at date t based on an information set $\mathcal{F}_t$ then, in the absence of arbitrage, the prices of assets satisfy the valuation formula:

$$C_t\left(g_{t+1}\right) = \mathrm{E}^{\mathbb{P}}\left[M_{t,t+1}\, g_{t+1} \,|\, \mathcal{F}_t\right], \tag{3.23}$$

where the SDF $M_{t,t+1}$ is a function of the updated information $\mathcal{F}_{t+1}$ and g_{t+1} is a payoff at date $t+1$. Gouriéroux and Monfort (2007) suggest an exponential affine form of the SDF:

$$M_{t,t+1} = e^{\alpha_{t+1} X_{t+1} + \beta_{t+1}}, \tag{3.24}$$

where X_{t+1} is a logarithmic return and the coefficients α_{t+1} and β_{t+1} are $\mathcal{F}_t$-measurable variables.

The conditional Esscher transform conforms to (3.24) by setting

$$\alpha_{t+1} = \theta^*_{t+1}, \quad \beta_{t+1} = -\log\big(\mathcal{M}_{X_{t+1}|\mathcal{F}_t}(\theta^*_{t+1})\big) - r. \tag{3.25}$$

The Esscher parameter θ^*_t in (3.13) depends on the conditional variance as well as the parameters of the Meixner distribution. Consequently, the dynamics of the process $\{\theta^*_t\}$ also depend on the dynamics of the historical variance process. The exponential affine SDF $M_{t,t+1}$ is, therefore, a function of θ^*_t, X_{t+1}, the Meixner parameters and the historical variance process embedded into the conditional parametric MGF.

The exponential affine SDF (3.25) is related to the Radon-Nikodym derivative in (3.7) via

$$M_{t,t+1} = \frac{\mathcal{L}_{t+1}}{\mathcal{L}_t}\, e^{-r}. \tag{3.26}$$

By the law of iterated expectations, it follows that the price of a $(T-t)$ - period payoff with a payoff function g_T is given by

$$C_t\left(g_T\right) = \mathrm{E}^{\mathbb{P}}\left[M_{t,t+1} \dots M_{T-1,T}\, g_T \,|\, \mathcal{F}_t\right]. \tag{3.27}$$

4 Model implementation

4.1 Estimation

The conditional distribution (3.4) of asset returns and property (2.4) of the Meixner density function permit an estimation of the parameters under the historical measure by means of the Maximum Likelihood method based on observed returns. As a first step, we use Quasi Maximum Likelihood Estimation (QMLE) to obtain the parameters $(\lambda, \alpha_0, \alpha_1, \beta_1)$ from a GARCH model with normal innovations and to extract the residuals $\hat{\varepsilon}_t$ from the mean equation $X_t = r + \lambda\sqrt{h_t} + \varepsilon_t$. Using the QMLE residuals $\hat{\varepsilon}_t$, we calculate the moment-based estimators of b and d by equating the sample skewness and kurtosis with (2.2). In a second step, we use the estimated parameters $(\hat{\lambda}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_1, \hat{b}, \hat{d})$ of the previous step as the initial values for an exact MLE of the GARCH model with Meixner innovations. The log-likelihood function is given by

$$L_T(x|\lambda, \alpha_0, \alpha_1, \beta_1, b, d) = -\frac{1}{2}\sum_{t=1}^{T} \log\left(\frac{h_t}{\sigma_{MD}^2}\right) + \sum_{t=1}^{T} \log f_{MD}\left(\frac{X_t - \mu_t}{\sqrt{h_t}}\,\sigma_{MD} + \mu_{MD}\right), \quad (4.1)$$

where T is the sample size and $f_{MD}(x; a, b, m, d)$ is the Meixner density function defined in (2.1). The Meixner moments μ_{MD} and σ_{MD} are defined in (2.2) and a and m are given by (2.6) and (2.7).

In GARCH-in-mean models, the information matrix is not block diagonal. Thus, asymptotic efficiency and consistent estimation of the parameters require that both the conditional mean and variance functions be estimated jointly. Asymptotic properties of the QML estimator in GARCH-type models have been investigated in Francq and Zakoïan (2004), who study the linear ARMA-GARCH case and prove strong consistency and asymptotic normality of the QML estimator under weak moment conditions. Similarly, Meitz and Saikkonen (2011) develop an asymptotic estimation theory for nonlinear AR(p)-GARCH(1,1) models in providing conditions comparable to those established by Francq and Zakoïan (2004). The most recent contributions to the asymptotic theory of QMLE in GARCH-in-mean models have been made by Conrad and Mammen (2015). They derive conditions that ensure consistency and asymptotic normality of the QMLE in the special case of a GARCH(1,1) process when the mean function does not grow too fast. These results ensure that the first-step QMLE estimates are consistent.

4.2 Option pricing

Option prices can be computed by simulations, which are performed either under $\mathbb{P}$ or under $\mathbb{Q}$. Due to positivity and tractability of the exponential affine SDF, we prefer the first approach for pricing European options in our Meixner GARCH models.

Under $\mathbb{P}$, the option price is given by (3.27), in conjunction with (3.25). The SDF generates the Radon-Nikodym derivative $\mathcal{L}_T = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ defined in (3.7). Therefore, the evaluation method can be based on the Monte Carlo approximation

$$e^{-rT}\frac{1}{N}\sum_{n=1}^{N}\mathcal{L}_T^{(n)}g(S_T^{(n)}) \longrightarrow C_t\left(g(S_T)\right), \quad \text{for } N \to \infty\,, \tag{4.2}$$

where $\mathcal{L}_T^{(n)}$ is the nth path of the Radon-Nikodym derivative calculated from the simulated volatility dynamics under $\mathbb{P}$, $S_T^{(n)}$ is the nth stock price path simulated under $\mathbb{P}$ and the payoff function is defined as $g(S_T^{(n)}) = (S_T^{(n)} - K)^+$ for call options and $g(S_T^{(n)}) = (K - S_T^{(n)})^+$ for put options with strike price K.

We employ the following Monte Carlo option pricing strategy for European put and call options where the stock or index returns follow the (T)GARCH models.

1. Estimate the model parameters by MLE using return data. Then generate the $(T-t)$ Meixner distributed random variables $\left(z_{t+1}^{(n)}, \ldots, z_T^{(n)}\right)$ with zero mean and unit variance using the rejection method (see Appendix) for each simulation n.

2. Simulate recursively $h_k^{(n)}$ and $X_k^{(n)}$ using the (T)GARCH specification in which $\varepsilon_k^{(n)} = z_k^{(n)}\sqrt{h_k^{(n)}}$, $k = t+1, \ldots, T$.

3. The nth path of $X_k^{(n)}$ and $h_k^{(n)}$ generates the stock price $S_T = S_t \exp\left(\sum_{k=t+1}^{T} X_k^{(n)}\right)$, the Esscher parameter $\theta_k^{(n)}$ by formula (3.13) and the Radon-Nikodym derivative $\mathcal{L}_T^{(n)}$ by formula (3.7).

4. From N simulated paths, we find the option price at time t as given by (4.2).

To reduce the Monte Carlo variance, we use the empirical martingale simulation (EMS) method proposed by Duan and Simonato (1998). Fengler et al. (2012) emphasize the application of the EMS scheme due to the following representation of the call option price

$$e^{-rT}\frac{1}{N}\sum_{n=1}^{N}\mathcal{L}_T^{(n)}(S_T^{(n)} - K)^+ = e^{-rT}\frac{1}{N}\sum_{n=1}^{N}(\mathcal{L}_T^{(n)}S_T^{(n)} - \mathcal{L}_T^{(n)}K)^+\,, \tag{4.3}$$

where the scheme is applied to both the discounted process $\{\mathcal{L}_t S_t\}$ and the process $\{\mathcal{L}_t\}$ under $\mathbb{P}$. Applying the EMS method in this way, we preserve the martingale property of these two processes. Hence, this strategy guarantees that the put-call parity holds for the simulated prices.

5 Empirical analysis on S&P 500 index options

5.1 Data

For our application, we consider S&P 500 index closing prices. The index price data consists of 5 044 daily observations taken from January 2, 1990, to January 5, 2010. Descriptive statistics of the daily logarithmic returns are given in Table 1.

We use European-style options on the S&P 500 index to test the models. To that end, we consider the closing prices of out-of-the-money (OTM) put and call options of each first

Wednesday of every month from January 6, 2010, to December 29, 2010. In general, OTM options are more liquid and more actively traded than in-the-money options. Option data and zero coupon data are downloaded from OptionMetrics. As option price we take the average of the bid and ask price. Following Barone-Adesi et al. (2008), options with time to maturity less than 10 days or longer than 360 days, zero traded volume, implied volatility larger than 70%, or prices less than \$0.3 are discarded, which yields a sample of 2571 options. Put and call option are represented in the sample in the ratio 60.5% and 39.5%, respectively.

The option data are categorized according to time to maturity (DTM), measured in calendar days, and moneyness (M), defined as the ratio of the strike price over the index price, K/S_t. A put option is said to be OTM if $M < 1$. The moneyness range of put options is divided into three intervals: $M < 0.9$ (656 options), $0.9 \leqslant M < 0.95$ (437 options) and $0.95 \leqslant M < 1$ (462 options). A call option is said to be OTM if $M > 1$. The moneyness range of call options is divided into the intervals: $1 < M \leqslant 1.05$ (430 options), $1.05 < M \leqslant 1.1$ (373 options) and $1.1 < M$ (213 options). An option is short maturity if $DTM < 60$ days (1520 options), medium maturity if $60 \leqslant DTM < 160$ days (720 options) and long maturity if $160 \leqslant DTM$ days (331 options). Average option prices range from \$7.39 for short maturity options to \$47.48 for long maturity options and from \$5.59 for deep OTM options to \$30.25 for options with moneyness close to 1.

5.2 Model fits and in-sample option pricing evaluations

We compare our Meixner models (MXN-GARCH and MXN-TGARCH) with the following benchmarks: (*i*) the models with normal innovations (GARCH and TGARCH) of Duan (1995); (*ii*) the models with shifted Gamma innovations (SG-GARCH and SG-TGARCH) of Siu et al. (2004); (*iii*) the models with NIG innovations (NIG-GARCH and NIG-TGARCH) which performs best according to Badescu et al. (2011, see their Table 2, p. 695). In all cases, the SDF is identified by means of the Esscher transform.[1]

The GARCH and TGARCH model parameters are estimated by exact MLE. For the models with Meixner and NIG innovations, we use the exact MLE estimation procedure described in Section 4.1. To compute the standard errors of the Meixner parameters a and m (NIG: a and b), we employ the Delta method. The shifted Gamma models are estimated based on the QMLE approach of Siu et al. (2004). Because the estimated parameters of the SG-GARCH and SG-TGARCH models coincide with the corresponding parameters of the GARCH and TGARCH models, we only report the parameters of models with normal innovations. The additional Gamma parameter estimates are $a = 24.938$, $b = 0.200$ for the SG-GARCH and $a = 24.302$, $b = 0.203$ for the SG-TGARCH model (using the notation in of Siu et al., 2004).

The results are shown in Table 2. The MXN-TGARCH and NIG-TGARCH models provide the best information criteria values and the highest value of the log-likelihood function among all models. The persistence is similar across all models and ranges from 0.9890 to 0.9965 as often found in the literature. Figure 2 exhibits the quantile-quantile plots of the estimated residuals of all models. As is visible, models with normal and shifted Gamma innovations do not properly capture the heavy-tailed returns distribution. The models with Meixner and NIG innovations provide a good fit thanks to their ability to capture negative conditional skewness.

To compare the option pricing performance of the models in-sample, we study European put and call options taken from the filtered data for January 6, 2010. These data consist of 212 options. The strike prices range from \$910 to \$1350. Time to maturity is from 10 to 346 days. The closing price is $S_0 = \$1137.14$. The annual risk-free rates for each time to maturity are taken from the zero coupon yields. Dividend yields are calculated from the put-call parity for each time to maturity. We employ the option pricing strategy described above and take the value of the estimated variance at January 5, 2010, as the initial value

[1] The Esscher transform and Duan's LRNVR coincide with normal innovations (Siu et al., 2004).

of the conditional variance for stock returns simulations. The option prices are estimated by simulating 20 000 paths. We measure the performance of the models in terms of the dollar root mean squared error (RMSE)

$$\mathrm{RMSE} = \sqrt{\frac{1}{M}\sum_{k=1}^{M}\left(C_k^{\mathrm{Model}} - C_k^{\mathrm{Market}}\right)^2}, \tag{5.1}$$

where M is the number of options in the sample.

Tables 3 and 4 present the in-sample pricing errors by moneyness and maturity. Figure 3 illustrates the pricing errors which are defined as the difference between the model and the market price for the TGARCH models.[2] As can be observed, the asymmetric models with a leverage effect in the variance equation strictly outperform the symmetric GARCH models for conditionally normal, NIG and Meixner innovations. Similar results are known in the literature (Härdle and Hafner, 2000; Christoffersen and Jacobs, 2004). The models with shifted Gamma innovations perform poorly, especially for long maturity options and for OTM put options. Intuition suggests that the conditional distribution of returns is not the shifted Gamma one. Because this distribution does not have infinite support, it does not fit the conditional distribution of returns properly.

Comparing the models, we see a visible performance impact after introducing the NIG and the Meixner distribution into the (T)GARCH specifications – see Table 3. Most noticeable improvements are in the pricing of puts and deep OTM calls. Across the different moneyess, NIG and the Meixner model are very close, with the NIG-TGARCH model being somewhat better for OTM puts and the Meixner for OTM calls. Looking at Table 4, which shows the results across the maturity classes, we see again that the MXN-TGARCH and the NIG-TGARCH models are superior and on par. MXN-TGARCH model does better for short and middle maturity puts, where as NIG-TGARCH prices better long maturity puts and short maturity calls. Figure 3 illustrates the appreciable advantage of the NIG and the

[2]Given the inferiority of GARCH models, we provide plots on the TGARCH models only.

MXN-TGARCH model. All models, however, tend to underprice put OTM options.

5.3 Out-of-sample option pricing evaluation

In this section, we study the out-of-sample performance for S&P 500 put and call options. We take the option data of the first Wednesday of each month, i.e., we consider 11 Wednesdays from February 3, 2010, to December 1, 2010. To evaluate the option prices, we use the in-sample parameters estimated under the historical measure at January 6, 2010. We follow the same pricing strategy as for the in-sample analysis.

Tables 5 and 6 show the out-of-sample pricing errors by moneyness and maturity categories. Overall, the results confirm the outcomes of the in-sample analysis. Asymmetric models with leverage effect outperform the symmetric GARCH specification with normal, shifted Gamma, NIG and Meixner innovations. The MXN-TGARCH model outperforms all the other models, for both put and long maturity call options, and exhibits considerably low pricing errors for long time to maturity put options, especially for deep OTM puts, but is closely followed by the NIG model. Generally, the normal and the SG model underprice put options and show a poor performance in pricing long maturity options. All models capture the OTM call option prices and the short maturity option prices well. Averaging over all moneyness and maturity buckets, the MXN-TGARCH provides the lowest pricing error (about $4.50). Finally, Figures 4 and 5 present the deviations of model prices from market prices for the TGARCH models as a function of moneyness and maturity, respectively. The plots underpin furthermore the properties of the Meixner and the NIG model to capture the term structure of option prices very well.

Overall, we may thus conclude that the fitting ability of the new Meixner model, both in-sample and out-of-sample, is at par with that of the NIG model, which is ranked best in Badescu et al. (2011). We conjecture that this is because both distributions share a number of similarities. They both belong to the class of semi-heavy tailed distributions and, in the present setting, share the same (and very small) number of free parameters, thereby avoiding overfitting. The Meixner distribution therefore is an attractive alternative to the NIG distribution for GARCH-based option pricing.

5.4 Stylized implied volatility patterns

As is observed in the in-sample and out-of-sample analysis, the TGARCH and MXN-TGARCH models exhibit marked pricing differences for long-dated options. In order to obtain insights into the drivers of this finding, we compare the implied volatility patterns of these models. We simulate both models using the same parameters and interpret the Meixner shape parameters b and d as free variables. We set the parameter b equal to -0.2 and -0.6, which corresponds to increasing the negative skewness of the Meixner density function; the parameter d is changed from to 0.1 and 0.9, which diminishes the peakedness of the Meixner density function (decreases both skewness and kurtosis – see (2.2)). Option prices are computed as described in Section 3.5. We simulate 20 000 Monte Carlo paths.

Figure 6 shows the implied volatility pattern across different moneyness for options with about one year to expiry. Here and in all plots, the TGARCH model with normal innovations serves as a benchmark. We see that b, the skewness parameter, has the most visible impact. Making the conditional density more skewed results into a substantially steeper implied volatility skew, while the impact of d is less visible. How is the term structure impacted? In Figure 7, we present the implied volatility term structure for OTM puts and in Figure 8 that of at-the-money options. Again the skewness parameter is most important, but the impact of d is also clearly discernible. Combinations of the Meixner parameters, which define more skewed and more heavy-tailed distributed innovations, imply a much slower decay of the skew for OTM options, while on the other hand implied volatility levels of at-the-money options increase more strongly. In this way, we can attribute the better performance of the

MXN-GARCH models directly to the shape parameters of the Meixner distribution.

6 Conclusions

We present GARCH-based models with Meixner innovations for option pricing. To derive the risk-neutral pricing measure, we make use of the conditional Esscher transform. We deduce that under the risk-neutral measure, logarithmic returns are still Meixner distributed, albeit with time-varying parameters. Our results allow us to estimate the model parameters from time-series observations and to price options by simulating the physical return process and the Radon-Nikodym derivatives process jointly. We compare the empirical performance of the considered models with others frequently cited in the literature, namely the (T)GARCH models with normal, shifted Gamma, and normal inverse Gaussian (NIG) innovations.

Our in-sample and out-of-sample study of S&P 500 index options demonstrates that the asymmetric variance dynamics along with asymmetric Meixner innovations capture skewness and excess kurtosis of asset returns very well. The Meixner model provides, along with the NIG model, the best fit to option prices. The outperformance is particularly large for long maturity options and out-of-the-money put options. The results of the in-sample analysis remain robust for the out-of-sample option pricing performance. By means of simulations, we provide supportive evidence that the better option pricing performance is due to conditional skewness and excess kurtosis of the Meixner distributed innovations.

Although the GARCH models with Meixner innovations perform remarkably well, there are still discrepancies for at-the-money options and out-of-the money call options. This could be due to a limitation in that the Esscher transform implies an exponential affine pricing kernel. Indeed, recent research suggests the nonlinear pricing kernels may in addition improve on the pricing accuracy, e.g., see Christoffersen et al. (2010) and Babaoglou et al. (2014). Alternatively, one could consider more general specifications of linear pricing kernels, e.g., see Christoffersen et al. (2013), Majewski et al. (2015) and Badescu et al. (2015). We leave this for future research.

A Meixner random variable generator

Grigoletto and Provasi (2008) suggest a fast generator of Meixner random variables by means of a rejection method. The approach assumes the existence of another density function $g(x)$ and a constant $c \geqslant 1$ such that $f_{MD}(x) \leqslant cg(x)$. A suitable choice is when $g(x)$ belongs to the Johnson translation family of unbounded functions

$$g(x;\xi_g,\lambda_g,\gamma_g,\delta_g) = \frac{\delta_g}{\lambda_g\sqrt{2\pi}\sqrt{u^2+1}} \exp\left(-\frac{1}{2}\left(\gamma_g + \delta_g \sinh^{-1}(u)\right)^2\right), \quad u = \frac{x-\xi_g}{\lambda_g}, \quad \text{(A.1)}$$

where $\xi_g, \lambda_g > 0, \gamma_g, \delta_g > 0$ are (real) parameters. The algorithm of Grigoletto and Provasi (2008) is as follows:

1. Determine the parameters $\xi_g, \lambda_g, \gamma_g, \delta_g$ assuming that the first four moments of the $f_{MD}(x)$ and $g(x)$ are equal.

2. Derive the constant c from the equation

$$c = \sup\left(\frac{f_{MD}(x;a,b,m,d)}{g(x;\xi_g,\lambda_g,\gamma_g,\delta_g)}\right). \quad \text{(A.2)}$$

3. Generate random variables $u \sim U(0,1)$, $z \sim N(0,1)$ and then $\bar{x}$ using the transform

$$\bar{x} = \xi_g + \lambda_g \sinh((z-\gamma_g)/\delta_g). \quad \text{(A.3)}$$

4. Accept $\bar{x}$ as a random variable from $MD(a,b,m,d)$ if the next inequality holds:

$$u \leqslant \frac{1}{c}\left(\frac{f_{MD}(\bar{x};a,b,m,d)}{g(\bar{x};\xi_g,\lambda_g,\gamma_g,\delta_g)}\right), \quad \text{(A.4)}$$

otherwise, repeat the algorithm from the previous step.

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Figures and Tables

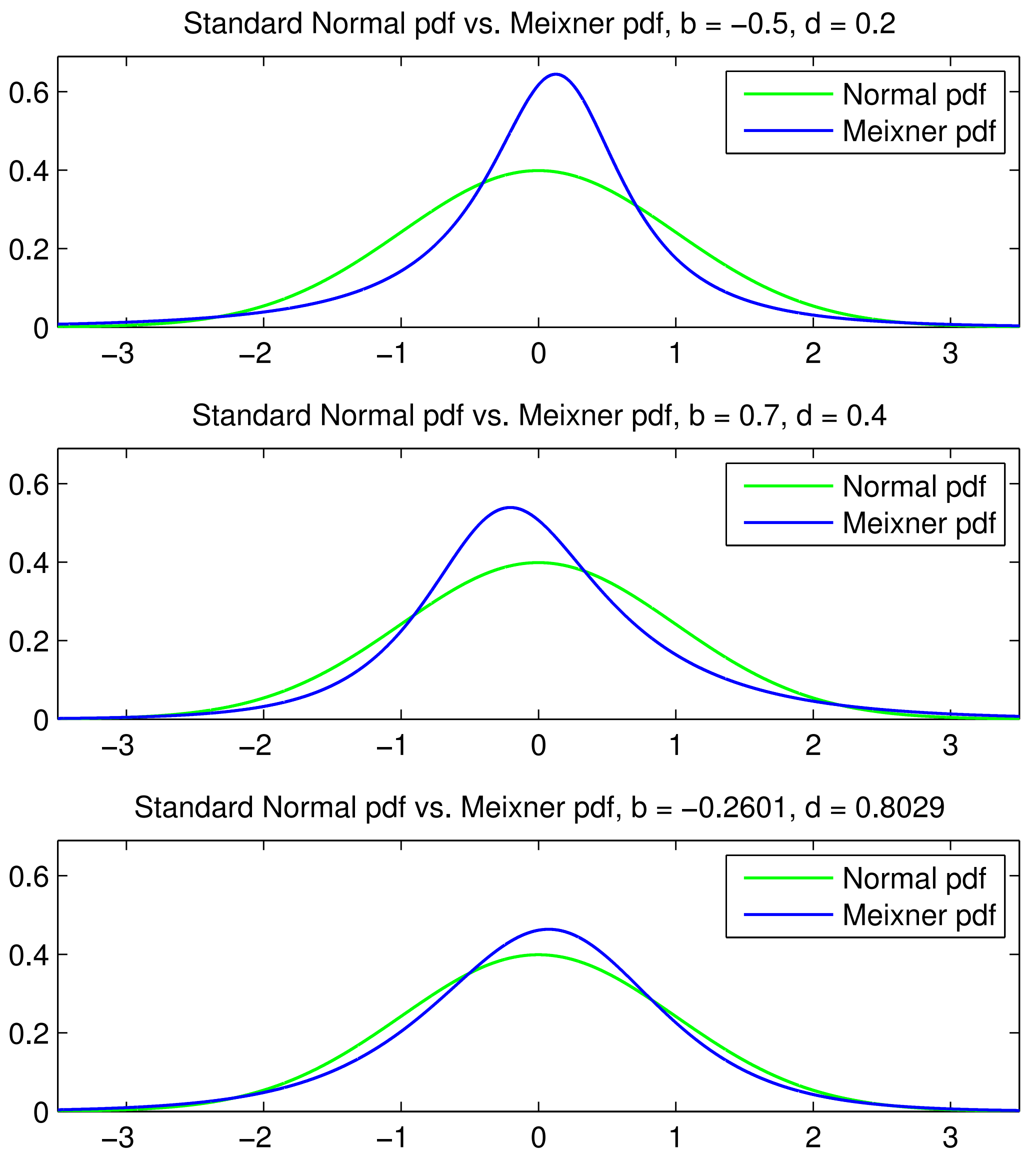

Figure 1: Comparison of the standard normal density and the Meixner density with zero mean and unit variance for different values of the shape parameters b and d. The bottom panel presents the Meixner density function with the parameters obtained from the maximum likelihood estimation of the S&P500 index price data in the MXN-GARCH model.

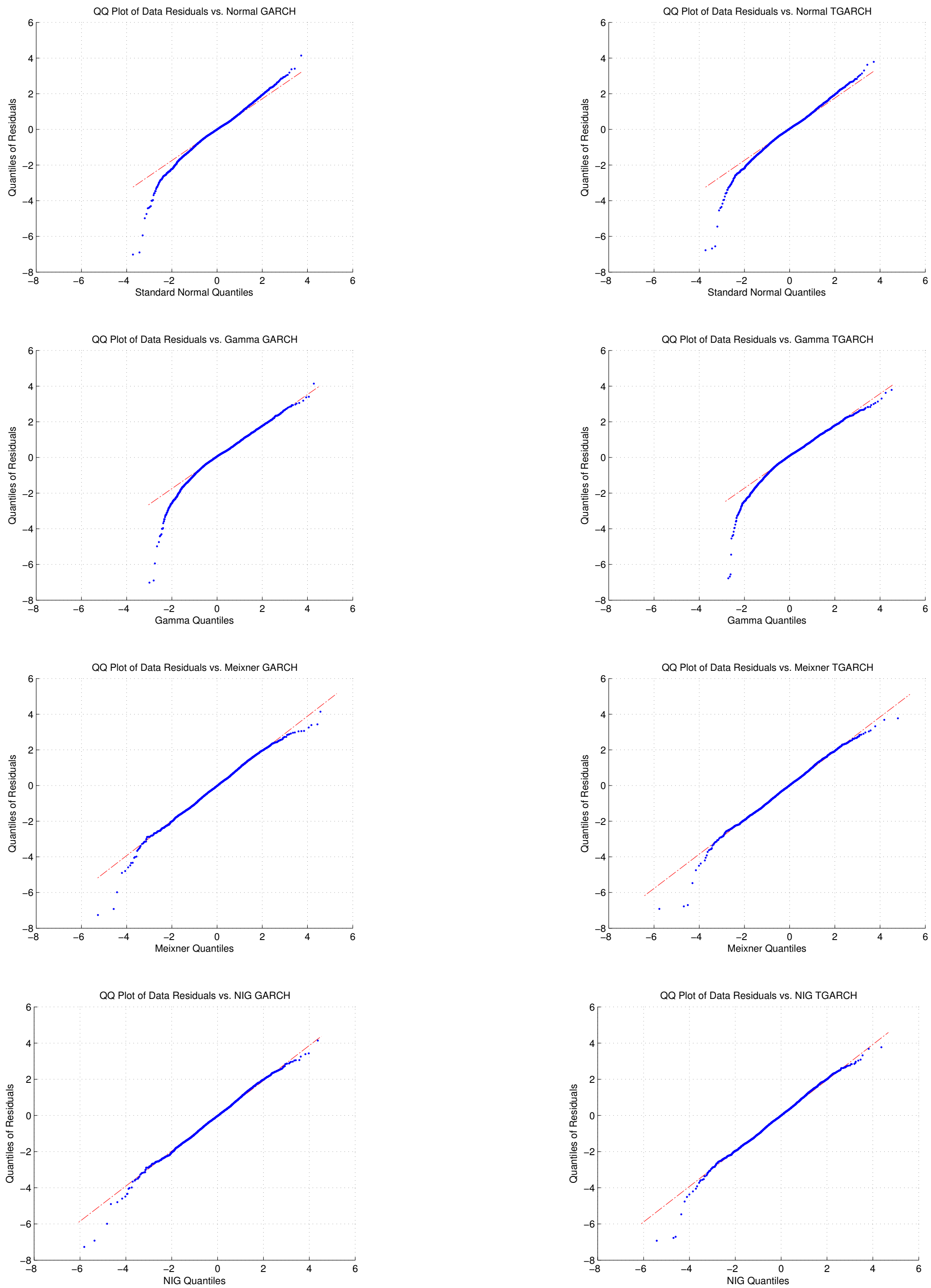

Figure 2: Quantile-quantile plots of estimated residuals in the (T)GARCH models with normal, shifted Gamma, Meixner and NIG innovations on S&P 500 index prices: January 2, 1990 - January 6, 2010.

Asset	Mean	Med.	Min.	Max.	Std.	Skew.	Kurt.
S&P 500	0.00023	0.00049	-0.0947	0.1096	0.0117	-0.1990	12.1709

Table 1: Summary statistics of daily index returns.

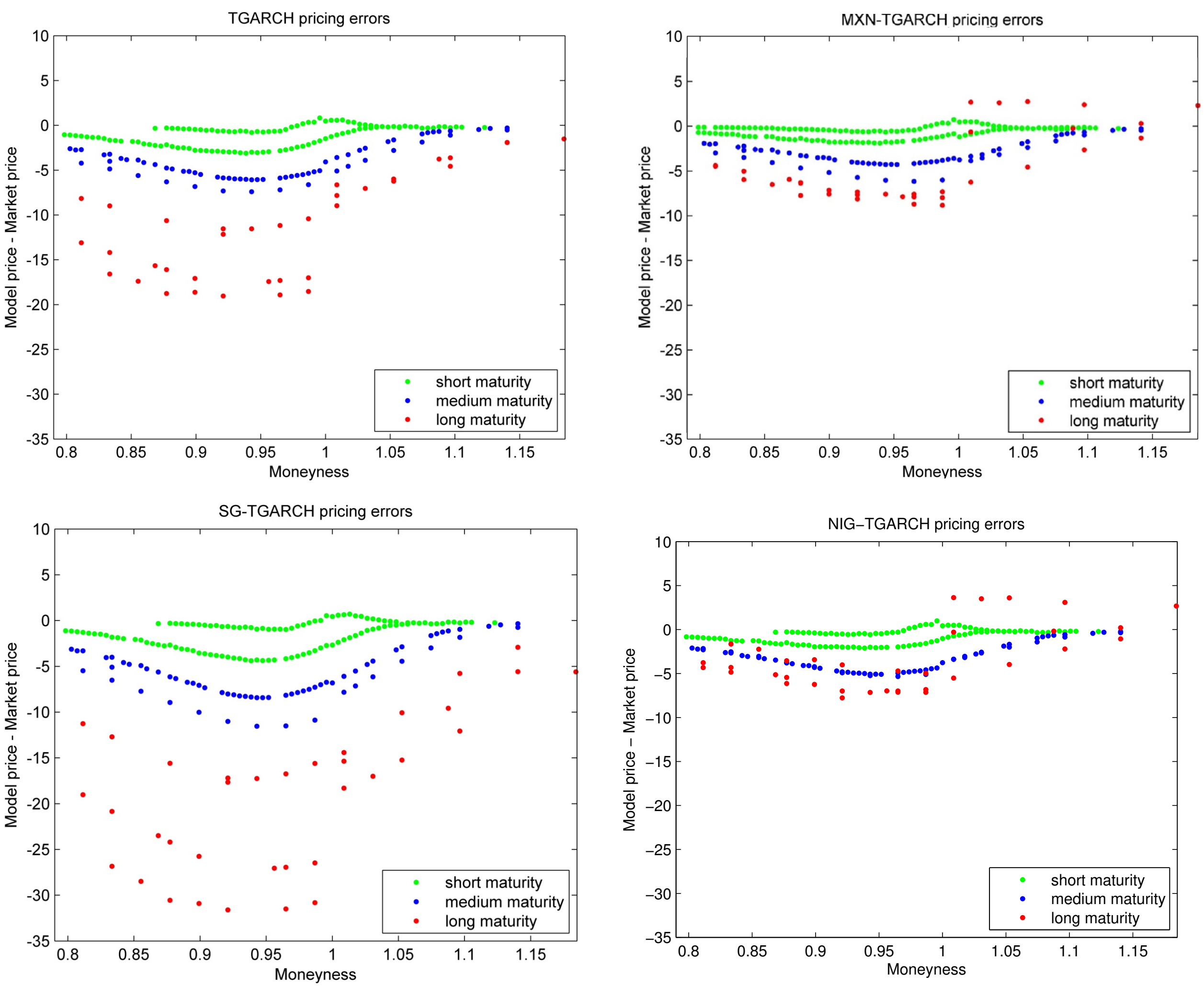

Figure 3: In-sample deviation of model prices from market prices. Left panel: TGARCH and SG-TGARCH. Right panel: TGARCH models with Meixner and NIG innovations.

Parameters	GARCH	TGARCH	MXN-GARCH	MXN-TGARCH	NIG-GARCH	NIG-TGARCH
Distr.	Normal	Normal	Meixner	Meixner	NIG	NIG
α_0	7.41×10^{-7}	1.09×10^{-6}	5.12×10^{-7}	8.74×10^{-7}	5.03×10^{-7}	8.63×10^{-7}
	(1.66×10^{-7})	(1.82×10^{-7})	(1.55×10^{-7})	(1.77×10^{-7})	(1.68×10^{-7})	(1.72×10^{-7})
α_1	0.0637	0.0018	0.0596	0.00067	0.0593	0.00057
	(6.48×10^{-3})	(1.61×10^{-3})	(7.19×10^{-3})	(1.90×10^{-4})	(7.22×10^{-3})	(1.38×10^{-4})
β_1	0.9306	0.9319	0.9368	0.9350	0.9371	0.9352
	(6.96×10^{-3})	(6.46×10^{-3})	(7.37×10^{-3})	(6.94×10^{-3})	(7.63×10^{-3})	(6.53×10^{-3})
λ	0.0669	0.0298	0.0532	0.0248	0.0534	0.0247
	(1.42×10^{-2})	(1.19×10^{-2})	(1.50×10^{-2})	(7.97×10^{-3})	(1.37×10^{-2})	(8.63×10^{-3})
γ_1		0.1105		0.1123		0.1124
		(1.07×10^{-2})		(1.22×10^{-2})		(1.21×10^{-2})
a			1.5649*	1.4264*	1.5366	1.7038
			(9.86×10^{-2})	(9.76×10^{-2})	(1.11×10^{-1})	(1.35×10^{-1})
b			-0.2601	-0.3432	-0.1751	-0.2527
			(8.64×10^{-2})	(7.52×10^{-2})	(4.83×10^{-2})	(6.25×10^{-2})
d			0.8029	0.9543	1.5068*	1.6480*
			(1.15×10^{-1})	(1.32×10^{-1})	(4.41×10^{-2})	(3.83×10^{-2})
m			0.1644*	0.2359*	0.1728*	0.2471*
			(4.49×10^{-2})	(5.68×10^{-2})	(4.65×10^{-2})	(5.77×10^{-2})
LL	16395.65	16463.33	16499.02	16553.35	16499.05	16553.75
AIC	-32783.30	-32916.67	-32986.04	-33092.70	-32986.11	-33093.91
BIC	-32757.20	-32884.04	-32946.89	-33047.02	-32946.94	-33048.23
Persist.	0.9943	0.9890	0.9964	0.9918	0.9965	0.9920

Table 2: Parameter estimates and their standard errors obtained by MLE. The NIG parameters a, b, m and d correspond to α, β, μ and δ in Badescu et al. (2011). Starred parameters are redundant and derived from the free estimated parameters; standard errors are obtained by means of the Delta method. Persistence is $\alpha_1 + \beta_1$ (GARCH), and $\alpha_1 + \frac{1}{2}\gamma_1 + \beta_1$ (TGARCH).

Model	Moneyness					
	Put options			Call options		
	< 0.90	$0.90 - 0.95$	$0.95 - 1.0$	$1.0 - 1.05$	$1.05 - 1.10$	$1.10 <$
GARCH	8.1110	7.8486	7.0086	**1.9835**	1.7082	3.0135
TGARCH	6.6177	6.8563	7.6286	3.2567	2.2374	2.0250
SG-GARCH	10.1530	10.2846	9.6743	3.1713	1.8458	2.6757
SG-TGARCH	9.9463	10.4802	11.9276	6.4299	4.7901	2.6926
MXN-GARCH	7.0829	7.2171	6.7856	2.1168	1.4322	2.5418
MXN-TGARCH	3.0947	3.8731	4.1933	2.0731	**1.4588**	**0.8873**
NIG-GARCH	7.7514	7.8579	7.87120	2.0612	1.4873	2.7433
NIG-TGARCH	**2.6973**	**3.6765**	**3.8680**	2.0193	1.4596	0.9440

Table 3: In-sample RMSE($) for different levels of moneyness on S&P 500 index prices: January 2, 1990 - January 6, 2010.

Model	Options	Maturity			Overall
		DTM < 60	60 < DTM < 160	160 < DTM	
GARCH	Put	2.1667	6.0735	16.6624	7.7341
	Call	1.2775	1.8160	3.7969	2.0581
	Overall	1.9048	5.0516	13.6890	6.3779
TGARCH	Put	1.8567	5.3345	15.1973	6.9878
	Call	0.4944	2.5133	5.5367	2.6594
	Overall	1.5269	4.5785	12.7390	5.8695
SG-GARCH	Put	2.6581	7.8057	21.8078	10.0624
	Call	1.3745	2.2211	5.2451	2.6493
	Overall	2.2942	6.4789	17.9432	8.2948
SG-TGARCH	Put	2.4922	7.4347	23.8475	10.6988
	Call	0.9089	4.0864	12.0554	5.4189
	Overall	2.0823	6.4976	20.5802	9.2282
MXN-GARCH	Put	1.9802	5.6050	15.1196	7.0430
	Call	1.1623	1.7713	3.7115	1.9829
	Overall	1.7396	4.6738	12.4478	5.8205
MXN-TGARCH	Put	**1.1400**	**3.8738**	7.0297	3.6745
	Call	0.4300	2.2008	**2.9147**	1.7190
	Overall	**0.9547**	**3.4010**	5.9487	3.1416
NIG-GARCH	Put	1.9042	5.8303	17.1563	7.8182
	Call	1.1657	**1.7353**	3.7038	1.9720
	Overall	1.6842	4.8484	14.0757	6.4355
NIG-TGARCH	Put	1.3100	4.1607	**5.5551**	**3.3709**
	Call	**0.3778**	2.0758	2.9901	**1.6866**
	Overall	1.0808	3.5931	**4.8316**	**2.9034**

Table 4: In-sample RMSE($) for different time to maturity and overall pricing errors.

Model	Moneyness					
	Put options			Call options		
	< 0.90	$0.90 - 0.95$	$0.95 - 1.0$	$1.0 - 1.05$	$1.05 - 1.10$	$1.10 <$
GARCH	7.6727	7.8646	7.0172	6.2046	6.2657	6.1143
TGARCH	5.9587	7.6050	7.4558	**5.0419**	4.4060	**3.5389**
SG-GARCH	8.9386	10.1455	8.5246	5.2437	6.1497	7.7342
SG-TGARCH	8.9641	11.2797	11.1405	6.9530	5.9287	4.1944
MXN-GARCH	5.9389	6.9348	6.6173	5.5184	6.3039	6.2532
MXN-TGARCH	**3.0984**	4.9469	**5.2749**	5.2066	**4.2522**	3.8881
NIG-GARCH	5.9452	7.0069	6.7750	5.5971	6.4021	8.3742
NIG-TGARCH	3.2624	**4.9361**	5.7177	5.3699	4.7013	4.6699

Table 5: Out-of-sample RMSE($) for different levels of moneyness on S&P 500 index prices: January 2, 1990 - January 6, 2010.

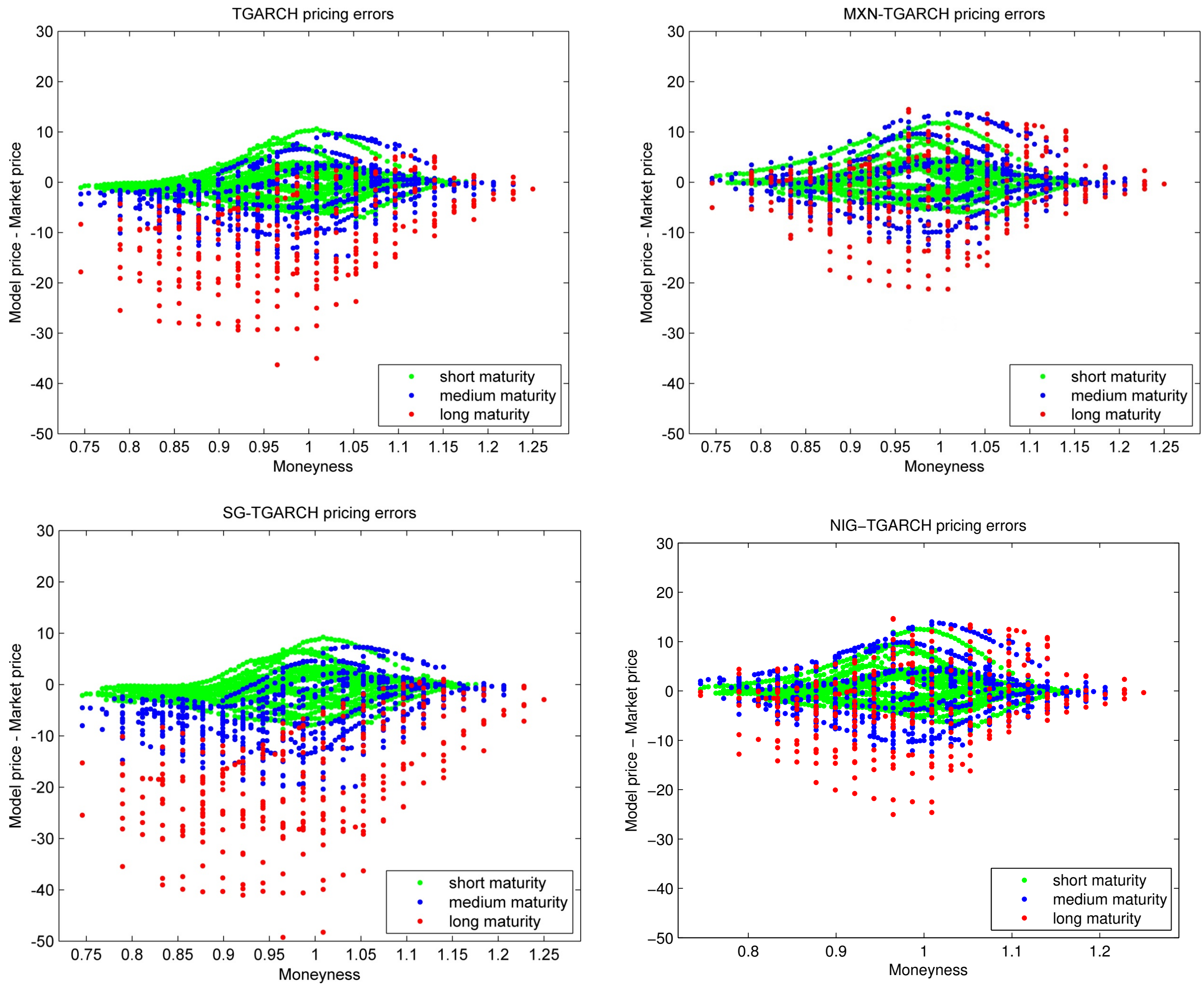

Figure 4: Out-of-sample deviation of model prices from market prices. Left panel: TGARCH with normal and SG innovations. Right panel: TGARCH with Meixner and NIG innovations.

Model	Options	Maturity			Overall
		DTM $<$ 60	$60 < \text{DTM} < 160$	160 $<$ DTM	
GARCH	Put	3.2800	7.6434	17.5410	7.5385
	Call	4.5422	6.2747	10.2632	6.2077
	Overall	3.7796	7.0362	14.7590	7.0380
TGARCH	Put	2.5242	6.5651	16.8923	6.9051
	Call	**3.5423**	**4.1812**	7.6009	**4.5229**
	Overall	2.9288	5.5780	13.5753	6.0685
SG-GARCH	Put	3.1680	9.1692	22.2071	9.1728
	Call	5.0504	7.1048	7.6368	6.1813
	Overall	3.9450	8.2677	17.3222	8.1128
SG-TGARCH	Put	3.0103	9.6091	25.9476	10.3144
	Call	3.7558	4.4852	13.3132	6.0763
	Overall	3.1002	7.6515	21.2839	8.8704
MXN-GARCH	Put	2.6278	6.6772	15.0119	6.4314
	Call	5.0490	7.6386	8.1688	6.4757
	Overall	3.6696	7.1432	12.4465	6.4491
MXN-TGARCH	Put	**2.4281**	**5.1298**	9.4963	**4.4583**
	Call	3.5497	5.3878	**6.3049**	4.5500
	Overall	**2.9234**	**5.2524**	**8.2674**	**4.4970**
NIG-GARCH	Put	2.7156	6.7465	15.1072	6.5040
	Call	5.0724	7.7531	8.4088	6.5719
	Overall	3.7325	7.2350	12.5809	6.5312
NIG-TGARCH	Put	2.4378	5.2830	**9.4637**	4.5843
	Call	4.1257	5.4294	6.6507	4.9868
	Overall	3.1465	5.3520	8.3326	4.7489

Table 6: Out-of-sample RMSE($) for different time to maturity and overall pricing errors.

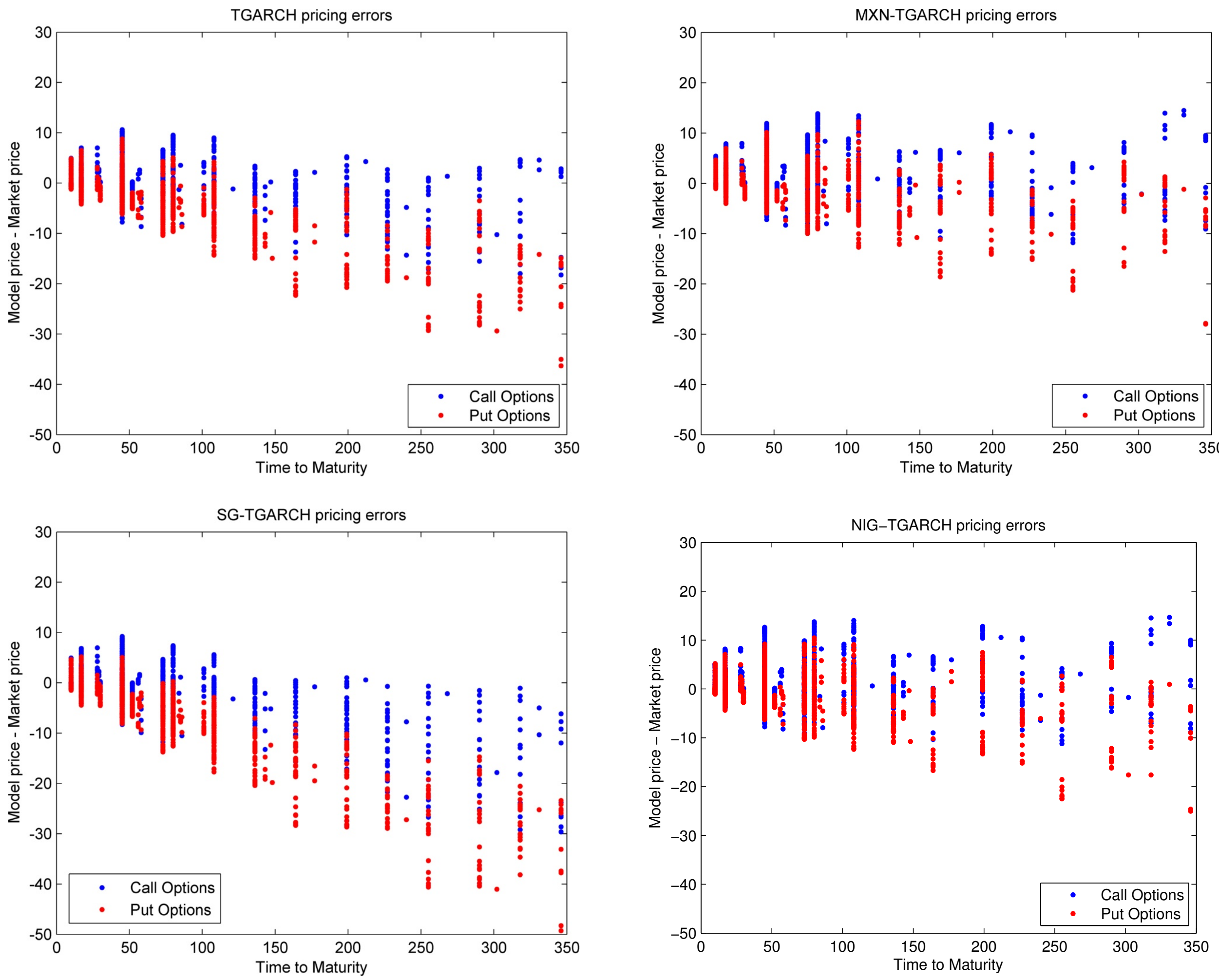

Figure 5: Out-of-sample deviation of model prices from market prices. Left panel: TGARCH with normal and SG innovations. Right panel: TGARCH with Meixner and NIG innovations.

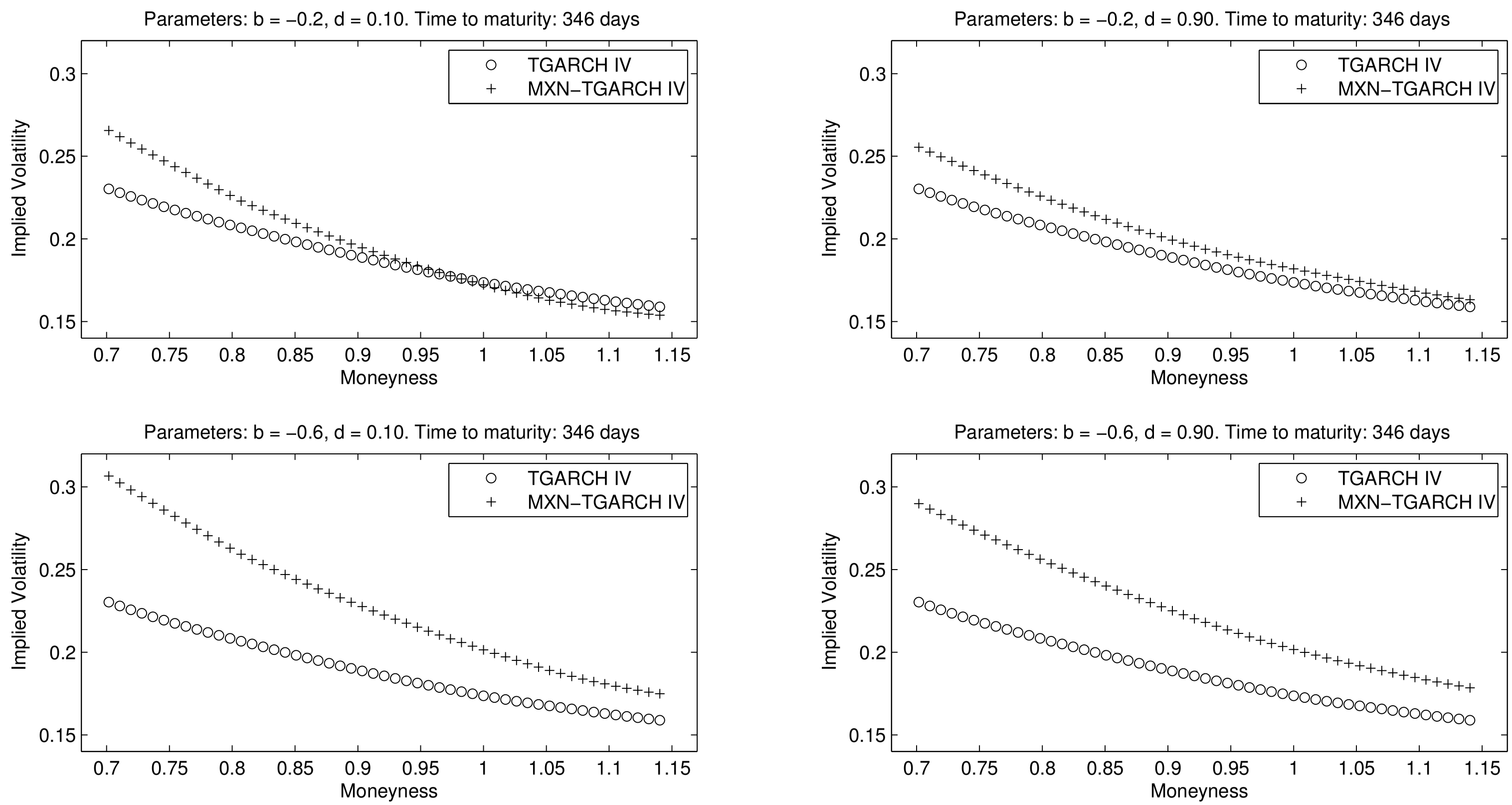

Figure 6: Implied Volatility MXN-TGARCH vs. TGARCH for the time to maturity 346 days.

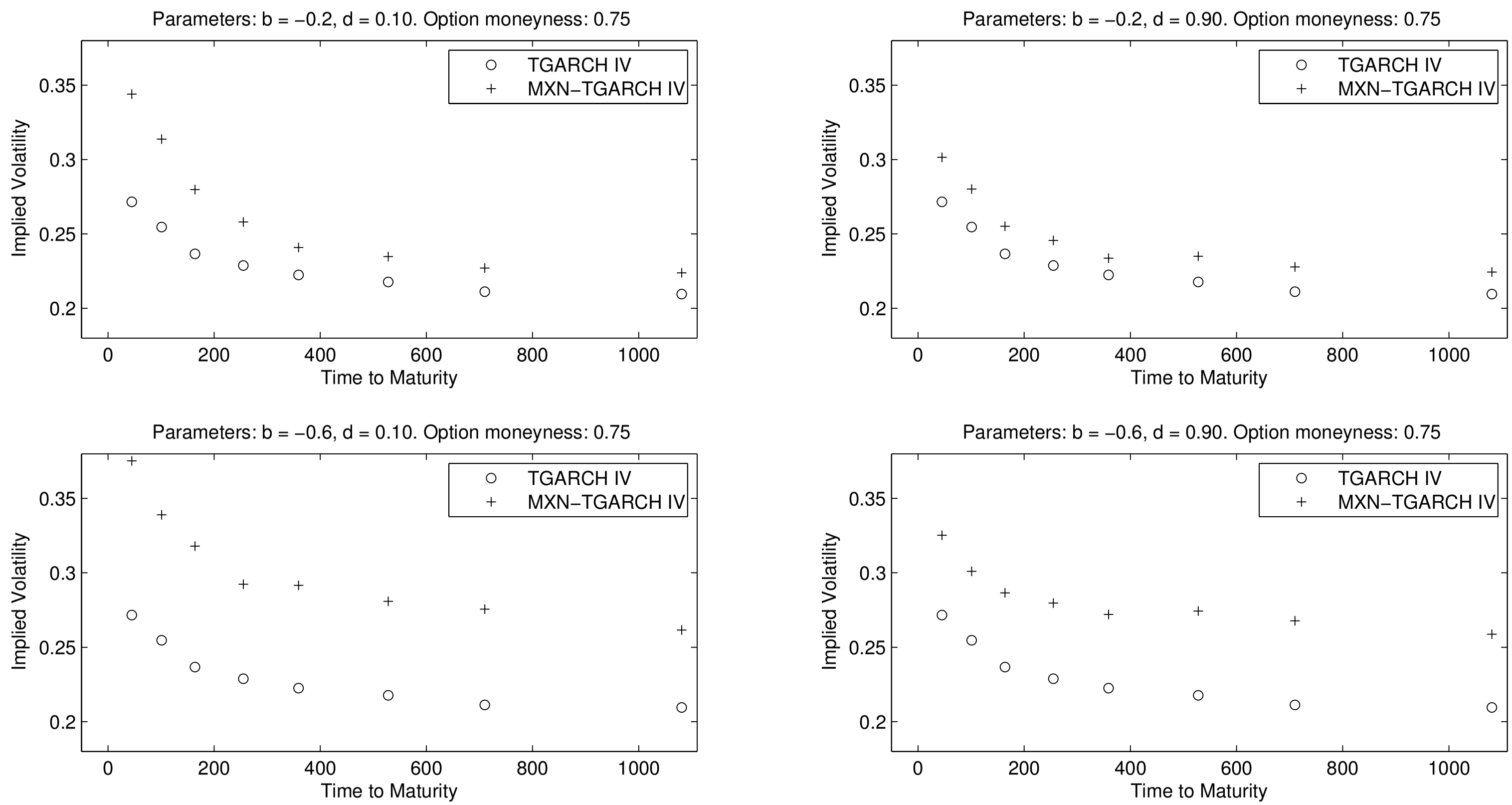

Figure 7: Implied Volatility MXN-TGARCH vs. TGARCH for the moneyness 0.75.

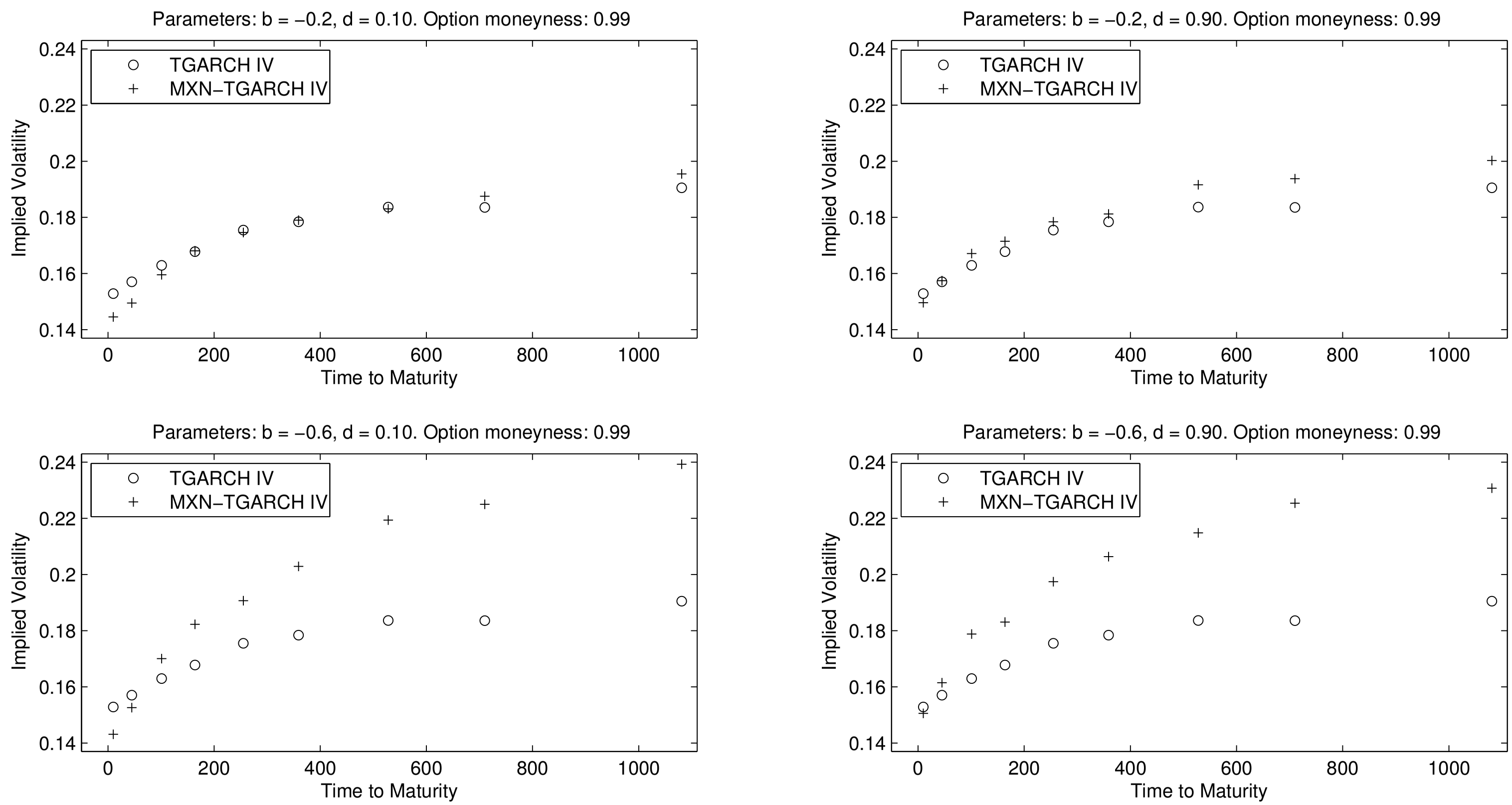

Figure 8: Implied Volatility MXN-TGARCH vs. TGARCH for the moneyness 0.99.