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Abstract

We investigate practical tests of market efficiency that are not subject to the joint-hypothesis problem inherent in tests that require the specification of an equilibrium model of asset prices. The methodology we propose simplifies the testing procedure considerably by reframing the market efficiency question into one about the existence of a local martingale measure. As a consequence, the need to directly verify the no dominance condition is completely avoided. We also investigate market efficiency in the large financial market setting with the introduction of notions of asymptotic no dominance and market efficiency that remain consistent with the small market theory. We obtain a change of numeraire characterization of asymptotic market efficiency and suggest empirical tests of inefficiency in large financial markets.

Keywords

Market efficiency, semimartingale, large financial markets, local martingale measure, asymptotic arbitrage.

JEL Classification

C12, G12, G14.

1 Introduction

An informationally efficient market is often understood to be one "in which prices always 'fully reflect' available information" (Fama, 1969, Page 383). While this description may be very helpful to the intuition, it leaves out at least one important ingredient: the probability measure relative to which prices are fully reflective of available information. If the probability measure assumed is the physical or statistical measure then this description may have more in common with the random walk hypothesis than market efficiency. Indeed, a slightly more rigorous description of market efficiency would require risk-adjusted asset prices to behave like martingales in the finite horizon, complete market setting; of course, the risk-adjustment may be swept up into an equivalent martingale measure Q, so that an alternative description of market efficiency, assuming finite horizon and market completeness, would require prices to evolve like martingales relative to a given information set and an equivalent martingale measure reflecting agent's preferences and risk tolerance. This description of market efficiency echoes Malkiel (1991, Page 211)'s take on the subject:

> A market is said to be efficient if it fully and correctly reflects all relevant information in determining security prices. Formally, the market is said to be efficient with respect to some information set, ϕ, if security prices would be unaffected by revealing that information to all participants. Moreover, efficiency with respect to an information set, ϕ, implies that it is impossible to make economic profits by trading on the basis of ϕ.

Hence, in an efficient (complete) market, risk-adjusted prices should be martingales, and it should be impossible to make economically significant profits by trading on the basis of available information. Moreover, since prices fully incorporate all available information at all times, there is no discrepancy between realized asset prices and prices implied by other non-price information. In other words, since such discrepancies are non-existent, there cannot exist trading strategies that perform *better* than buying and holding individual traded assets.

This latter intuition informs the rigorous characterization of market efficiency proposed in (Jarrow & Larsson, 2012, Theorem 3.2). The authors define a price process S as being efficient relative to a reference information set if an economy $\mathcal{E}$, determined by agents beliefs, endowments, and preferences, and a consumption good price index may be found such that S corresponds to equilibrium asset prices in $\mathcal{E}$. From this basic definition, they obtain characterizations in terms of the existence of equivalent martingale measures and in terms of the joint satisfaction of the no free lunch with vanishing risk (NFLVR) condition and the no dominance (ND) condition. NFLVR is an "absence of arbitrage" condition that ensures the existence of an equivalent local martingale measure for S, whereas ND imposes an optimality condition on asset prices.

One of the benefits of characterizing market efficiency in terms of NFLVR and ND is that the equivalent (risk-neutral) separating probability measure is taken out of the definition of market efficiency. From the empirical point of view, a way of testing efficiency without running into the joint-hypothesis issue is suggested since both NFLVR and ND are expressed entirely in terms of the physical or statistical probability measure. The joint hypothesis problem essentially describes the unquantifiability of the misspecification error incurred by specifying an equilibrium asset pricing model or stochastic discount factor as reference for testing market efficiency.

In the present work, we further this line of research by obtaining additional characterizations of market efficiency that have the advantage of simplifying empirical tests. In principle, the no dominance condition has to be verified for each asset, so that for a market with a large number of assets testing the ND requirement may prove to be impractical. Our first insight into the problem comes from the fact that, under the condition of no unbounded profit with bounded risk (NUPBR), the i-th asset S^i in a market with n distinct assets satisfies no dominance if and only if the n-dimensional vector of asset prices S, expressed in units of $(\gamma + (1-\gamma)S^i)$, $0 < \gamma < 1$, does not violate the no arbitrage (NA) condition. Moreover, not only are convex portfolios of undominated assets necessarily undominated (Delbaen & Schachermayer, 1997), but the converse is also true (Corollary 2.0.1). Combining, this insights with the fact that NUPBR remains invariant to a change of numeraire, we obtain a characterization of market efficiency in terms of NFLVR for S expressed in units of the market portfolio (Proposition 2.4). From an empirical standpoint, this characterization obviates the need for direct verification of the no dominance condition.

This reformulation also allows us to employ existing empirical techniques for testing market efficiency. The empirical tests devised in (Jarrow et al., 2012) and (Hogan et al., 2004) were originally intended to test for (statistical) arbitrage strategies. The absence of arbitrage is not sufficient for market efficiency. It is in fact possible for a given strategy to not be an arbitrage while violating the ND condition for one or more assets. But since a violation of the NA requirement for S expressed in units of the market portfolio is equivalent to a violation of the ND condition for one or more assets, we are able to repurpose the statistical arbitrage tests of Hogan et al. (2004) to perform simultaneous verifications of violations of the ND condition for all assets. And since the NA condition is equivalent to ND for the zeroth asset, both the NA and ND conditions can be handled within a single test.

We conclude our study of market efficiency by introducing notions of no dominance and market efficiency to the large financial market setting of Kabanov & Kramkov (1994, 1998); Cuchiero et al. (2015). We refer to these notions as asymptotic no dominance (AND) and asymptotic market efficiency (AME), respectively. The tests of market efficiency we propose in the standard small market setting assume the investment horizon is infinite, $\mathbb{R}_+$, so that they may be most appropriately used to test longer horizon strategies. The large financial market setting makes it possible to study fixed horizon strategies under the assumption that the number of asset in the market tends to infinity, much like in the arbitrage pricing theory (APT) framework of Ross (1976a,b). We obtain a further change of numeraire characterization and suggest tests for violations of asymptotic market efficiency.

2 Efficiency in standard markets

We take as given a filtered probability basis $\mathcal{B} := (\Omega, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, P)$ satisfying the usual conditions. Defined on $\mathcal{B}$ we assume an n-dimensional semimartingale $S := (S_t)_{t\geq 0}$ representing the price process of n assets. We will refer to the pair $(S, \mathcal{B})$ as a market.

Let $\lambda > 0$; we define λ-admissible strategies in the usual manner, i.e., n-dimensional predictable processes H such that the stochastic integral $H \bullet S$ is well-defined, $(H \bullet S) \geq -\lambda$, $\lim_{t\to\infty}(H \bullet S)_t = (H \bullet S)_\infty$ exists, and $H_0 = 0$. A strategy is said to be admissible if there exists $\lambda > 0$ such that it is λ-admissible. An *arbitrage* is an admissible strategy H such that $(H \bullet S)_\infty \geq 0$ almost surely and $(H \bullet S)_\infty > 0$ holds with positive probability. A market $(S, \mathcal{B})$ is said to satisfy the *no arbitrage* condition (NA) if it is devoid of arbitrage strategies. For admissible strategies, the random variable $(H \bullet S)_\infty$ is referred to as the terminal value or payoff of strategy H. The payoff space of 1-admissible strategies is denoted $\mathcal{K}_1$. The market $(S, \mathcal{B})$ is said to satisfy the no unbounded profit with bounded risk condition (NUPBR) if $\mathcal{K}_1$ is bounded in probability, i.e, bounded in the set of finite valued random variables $L^0(\mathcal{B})$. Now, if every sequence $f_n \in \mathcal{K}_1$ satisfying $\|f_n \wedge 0\|_\infty \to 0$ must also satisfy $f_n \xrightarrow{P} 0$ then the market $(S, \mathcal{B})$ is said to satisfy the no free lunch with vanishing risk condition (NFLVR).

The NFLVR condition is a strengthening of the NA condition. As a matter of fact, NFLVR is necessary and sufficient for both NA and NUPBR to hold (Kabanov, 1996, Lemma 2.2). In the general unbounded semimartingale case, the NFLVR condition is equivalent to the existence of a probability Q equivalent to P such that the components of S are stochastic integrals of a predictable process with respect to a local martingale. The measure Q is said to be an equivalent σ-martingale measure for S (Delbaen & Schachermayer, 1999, 1.1 Theorem). In the case of locally bounded S, Q is a local martingale measure for S (Delbaen & Schachermayer, 1994, Corollary 1.2). This is also the case for non-negative asset prices, i.e. NFLVR is equivalent to the existence of a probability measure Q, equivalent to P, such that S is a Q local martingale if $S \geq 0$ (Ansel & Stricker, 1994, Corollary 3.5).

The basic intuition of an efficient market relative to an information set $(\mathcal{F}_t)_{t\in[0,T]}$ (at least in the complete markets, finite horizon case) is that risk-adjusted prices evolve over time like $\mathcal{F}_t$-martingales. As a result, current prices represent the best prediction of the future behavior of risk-adjusted prices. This is the same as saying that any attempt, in the form of a trading strategy based on current information, to achieve a better outcome, in the form of superior risk-adjusted returns, than simply buying and holding the individual traded assets would ultimately prove to be unsuccessful. Note the close relationship between the available information set and the set of admissible trading strategies. The available information set uniquely determines the set of admissible trading strategies and vice versa. Hence, in describing market efficiency, we may speak of trading strategies rather than information set. Indeed, provided asset prices exist, an alternative characterization of markets efficiency may be stated in terms of the *no dominance* (ND) condition.

2.1 Definition (Jarrow & Larsson (2012)) *Given and n-dimensional S vector representing asset prices, the i-th component S^i is undominated on the time horizon $[0,T]$, $T<\infty$, if there is no admissible strategy H such that*

$$P((H\bullet S)_T \geq S^i_T - S^i_0) = 1 \quad \textit{and} \quad P((H\bullet S)_T > S^i_T - S^i_0) > 0. \tag{2.1}$$

The market $(S,\mathcal{B})$ is said to satisfy ND if $S^i, 0\leq i<n$, is undominated. We will assume in the current setting that the investment horizon is the positive real line, in contrast to the finite horizon setup analyzed in (Jarrow & Larsson, 2012). This modeling choice is important, since, in this section, we are primarily interested in devising tests of market efficiency that hold asymptotically as the investment horizon approaches infinity. We assume that prices have been rescaled so that $S^i_0 = 1$ for $0\leq i<n$. We also assume that $H_0 = 0$ for all admissible strategies. Hence, as a slight modification of the definition of ND given above, we will say that the i-th asset is undominated if for all admissible strategies H, $P((H\bullet S)_\infty \geq S^i_\infty - 1) = 1$ implies

$P((H \bullet S)_\infty = S^i_\infty - 1) = 1$. We now state the definition of market efficiency in our setting as follows:

2.2 Definition (Market efficiency) *Let* $(S, \mathcal{B})$ *be a market carried on the filtered probability basis* $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{F}, P)$. *It is said to be efficient if it satisfies NFLVR and ND.*

The above definition adapts the second characterization of market efficiency in (Jarrow & Larsson, 2012, Theorem 3.2(ii)) to our setting where the time horizon is infinite. Hence, a market is efficient if both NFLVR and ND are satisfied. Here, our objective is to derive equivalent characterizations of market efficiency that may be more suitable for empirical analysis. In the sequel, we will assume that the vector of prices S is expressed in terms of the asset occupying the zeroth position, so that $S^0_t = 1$ for $t \geq 0$, and that all other assets prices are non-negative so that $S^i_t \geq 0$ for all $t \geq 0$ and $0 \leq i < n$. Let $0 < \gamma < 1$ and denote $\tilde{S}^{\gamma,i}$ the $n+1$ dimensional vector obtained by appending $S^{\gamma,i} := (\gamma + (1-\gamma)S^i)$ to the end of S, i.e. $\tilde{S}^{\gamma,i} := (S, S^{\gamma,i})$. Now set

$$Z^{\gamma,i} := \tilde{S}^{\gamma,i}(S^{\gamma,i})^{-1}. \tag{2.2}$$

That is, $Z^{\gamma,i}$ expresses the price process S in units of the convex portfolio consisting of the zeroth asset and the i-th asset, $S^{\gamma,i}$.

2.1 Proposition *If the market* $(S, \mathcal{B})$ *is efficient then* $(Z^{\gamma,i}, \mathcal{B})$ *admits a local martingale measure for all* $0 \leq i < n$ *and* $0 < \gamma < 1$.

Proof. Recall that a σ-martingale measure, under which $Z^{\gamma,i}$ may be expressed as a stochastic integral with respect to a local martingale, coincides with a local martingale measure for non-negative $Z^{\gamma,i}$ (Ansel & Stricker, 1994, Corollary 3.5). Hence, it is only required to demonstrate NFLVR, which in turn is equivalent to both NA and NUPBR (Delbaen & Schachermayer, 1994, Corollary 3.8). Suppose $(S, \mathcal{B})$ satisfies NFLVR and ND while $(Z^{\gamma,i}, \mathcal{B})$ fails to satisfy NA for some i and $0 < \gamma < 1$, so that there exists an admissible strategy H for $Z^{\gamma,i}$ such that $(H \bullet Z^{\gamma,i})_\infty$ is non-negative and strictly positive with positive probability. We will argue as in (Delbaen & Schachermayer, 1995, Theorem 11). By rescaling H it may be assumed that H is 1-admissible for $Z^{\gamma,i}$. Consider the gain process

$$Y := (1-\gamma)^{-1}(S^{\gamma,i}(H \bullet Z^{\gamma,i} + 1) - 1). \tag{2.3}$$

Since $S^{\gamma,i}$ is strictly positive and H is 1-admissible for $Z^{\gamma,i}$, we have that $Y_t \geq -(1-\gamma)^{-1}$ for $t \geq 0$. Also, because $H_0 = 0$ and $S^{\gamma,i}_0 = 1$, we have that $Y_0 = 0$. Since H is an arbitrage for $Z^{\gamma,i}$, we have that Y_∞ is at least as great as $(1-\gamma)^{-1}(S^{\gamma,i}_\infty - 1) = S^i_\infty - 1$ with the inequality holding strictly with positive probability. ND will be violated for S^i if Y is representable as a stochastic integral with respect to S. This follows from an application of Itô's integration by parts formula. Write $H =: (H^a, H^b)$ where H^a denotes the first n components of H and H^b its $n+1$-st (last) component. Then

$$\begin{aligned}(1-\gamma)Y + 1 &= H \bullet S^{\gamma,i} \bullet Z^{\gamma,i} + H \bullet Z^{\gamma,i} \bullet S^{\gamma,i} + H \bullet [S^{\gamma,i}, Z^{\gamma,i}] + S^{\gamma,i} \\ &= H \bullet \tilde{S}^{\gamma,i} + S^{\gamma,i} \\ &= H^a \bullet S + H^b \bullet S^{\gamma,i} + S^{\gamma,i}.\end{aligned}$$

So that $Y = (1-\gamma)^{-1} H^a \bullet S + (H^b + 1) \bullet S^i$, which may be expressed as $K \bullet S$, where $K = (1-\gamma)^{-1} H^a + I^{n,b,i}$ and $I^{n,b,i}$ is the n dimensional vector which is zero everywhere except in the i-th position where it is equal to H^b+1. Hence, S^i is dominated by the $(1-\gamma)^{-1}$-admissible strategy K. This contradicts the efficiency of $(S, \mathcal{B})$.

We note that the result that the NUPBR condition is invariant to a change of numeraire in the finite horizon setting is proved in (Takaoka & Schweizer, 2014, Proposition 2.7 (ii)) using functional analytic methods. Here, we establish the claim using more elementary arguments. To that end, suppose $(S, \mathcal{B})$ is efficient while $(Z^{\gamma,i}, \mathcal{B})$ fails to satisfy the NUPBR condition. In that case there exists a sequence $(H^m)_{m\geq 1}$ of 1-admissible strategies for $Z^{\gamma,i}$ and $\beta > 0$ such that given $N \in \mathbb{N}$ if m is sufficiently large then $P((H^m \bullet Z^{\gamma,i})_\infty > N) > \beta$. Denote

$$Y^m := S^{\gamma,i}(H^m \bullet Z^{\gamma,i} + 1) - 1.$$

It is easily verified that $Y^m_0 = 0$ and $Y^m_t \geq -1, t \geq 0$. Moreover, $Y^m_\infty \geq \gamma((H^m \bullet Z^{\gamma,i})_\infty + 1) - 1$ so that $(Y^m_\infty)_{m\geq 1}$ is an unbounded sequence in $L^0(\mathcal{B})$. Indeed, for $N \in \mathbb{N}$, $P(Y^m_\infty > N) \geq P((H^m \bullet Z^{\gamma,i})_\infty > \gamma^{-1}(N+1) - 1)$, which for sufficiently large m exceeds β. NUPBR for $(S, \mathcal{B})$ will be violated as soon as Y^m is shown to be representable as a stochastic integral with respect to S. But this follows, as in the previous paragraph, from Itô's integration by parts formula. □

We now establish the converse to the previous claim.

2.2 Proposition *If $(Z^{\gamma,i}, \mathcal{B})$ admits a local martingale measure for every $0 < \gamma < 1$ and $0 \leq i < n$ then $(S, \mathcal{B})$ is efficient.*

Proof. Suppose $(Z^{\gamma,i}, \mathcal{B})$ admits a local martingale measure for every i and $0 < \gamma < 1$ while $(S, \mathcal{B})$ fails to satisfy NUPBR. Then there is H^m 1-admissible for S and $\beta > 0$ such that for sufficiently large m, $P((H^m \bullet S)_\infty > N) > \beta$ for all $N \in \mathbb{N}$. Consider

$$Y^m := (S^{\gamma,i})^{-1}(H^m \bullet S + 1) - 1.$$

Y_t^m is well-defined because $S_t^{\gamma,i}$ is strictly positive; $Y_t^m \geq -1$ because H^m is 1-admissible for S; and $Y_0^m = 0$ because $H_0^m = 0$ and $S_0^{\gamma,i} = 1$. We note that the existence of a local martingale measure for $Z^{\gamma,i}$ is equivalent to NFLVR. Under NFLVR $(H \bullet Z^{\gamma,i})_\infty$ exists and is finite-valued for admissible strategies (Delbaen & Schachermayer, 1994, Theorem 3.3). In particular $Z_\infty^{\gamma,i}$ and therefore $(S_\infty^{\gamma,i})^{-1}$, for all all i, is well-defined and finite valued. Note that $(S_\infty^{\gamma,i})^{-1}$ cannot be zero with positive probability since this would imply that $P(S_\infty^i = \infty) > 0$, which would contracdict the almost sure finiteness of $Z_\infty^{\gamma,i}$. Hence, $0 < (S_\infty^{\gamma,i})^{-1} \leq \gamma^{-1}$, almost surely, so that there is $c > 0$ sufficiently small such that $P((S_\infty^{\gamma,i})^{-1} \leq c) \leq \beta/2$. Hence,

$$P(Y_\infty^m > N) > P((H^m \bullet S)_\infty > c^{-1}(N+1) - 1) - P((S_\infty^{\gamma,i})^{-1} \leq c),$$

which for sufficiently large m is larger than $\beta/2$. That is $(Y_\infty^m)_{m\geq 1}$ is unbounded in $L^0(\mathcal{B})$.

Now let $K^m := (H^m, 0)$ denote the $n+1$-dimensional predictable process obtained by appending 0 to H^m. It is easily seen that $K^m \bullet \tilde{S}^{\gamma,i} = H^m \bullet S$. So that Y^m may be written alternatively as $(S^{\gamma,i})^{-1}(K^m \bullet \tilde{S}^{\gamma,i} + 1) - 1$. By Itô's integration by parts formula and the fact that $Z^{\gamma,i} = (S^{\gamma,i})^{-1}\tilde{S}^{\gamma,i}$, we have $Y^m = (K^m + I^{n+1,0}) \bullet Z^{\gamma,i}$, where $I^{n+1,0}$ denotes the $n+1$ dimensional vector with zeros everywhere except in the zeroth position where there is a 1. Thus, $(Y_\infty^m)_{m\geq 1}$ is generated by 1-admissible strategies for $Z^{\gamma,i}$ and therefore constitutes a violation of the NUPBR condition for $Z^{\gamma,i}$.

We note that the NA condition is simply the ND condition for the zeroth asset, so that demonstrating ND for $0 \leq i < n$ is all that is required. To that end, suppose there is an i and a c-admissible, $c > 0$, strategy H for S such that $(H \bullet S)_\infty \geq S_\infty^i - 1$ holds, almost surely, with the inequality holding strictly on a set of positive probability. Observe that this implies that

$$(1-\gamma)(H \bullet S)_\infty \geq (1-\gamma)(S_\infty^i - 1) = S_\infty^{\gamma,i} - 1 \tag{2.4}$$

holds almost surely with the inequality holding strictly on a set with positive probability. Set $K := (H, 0)$ and note that $H \bullet S = K \bullet \tilde{S}^{\gamma,i}$ for any $0 < \gamma < 1$; fix one such γ and define

$$Y := (S^{\gamma,i})^{-1}((1-\gamma)K \bullet \tilde{S}^{\gamma,i} + 1) - 1.$$

It is easily seen that $Y_0 = 0$ and easily verified that $Y_t \geq \gamma^{-1}(1-\gamma)(1-c)$ for $t \geq 0$. It follows from (2.4) that $P(Y_\infty \geq 0) = 1$ and $P(Y_\infty > 0) > 0$. It follows by the stochastic integration by parts formula that $Y = ((1-\gamma)K + I^{n+1,0}) \bullet Z^{\gamma,i} =: J \bullet Z^{\gamma,i}$. Hence, J constitutes a violation of the NA condition for $Z^{\gamma,i}$. This completes the demonstration. $\square$

The previous two Propositions may be summarized as follows:

2.3 Proposition *The market $(S, \mathcal{B})$ is efficient if and only if $(Z^{\gamma,i}, \mathcal{B})$ admits an equivalent local martingale measure for each $0 \leq i < n$ and $0 < \gamma < 1$. In particular, under NUPBR, the i-th asset S^i is undominated if and only if $(Z^{\gamma,i}, \mathcal{B})$ satisfy NA for all $0 < \gamma < 1$. Moreover, $(S, \mathcal{B})$ satisfies NUPBR if and only if $(Z^{\gamma,i}, \mathcal{B})$ satisfies NUPBR.*

It is easy to see that if Proposition 2.3 holds for one $\gamma \in (0,1)$ then it must hold for all $0 < \gamma < 1$. Hence, we may restate the claim of that Proposition using equally weighted portfolios of the numeraire asset and the i-th asset. These results make it somewhat easier to test for efficiency by leveraging econometric techniques designed for testing arbitrage and unbounded profit opportunities as opposed to attempting to test for the no dominance condition directly. Still a market with n assets would require $n+1$ tests to verify efficiency. The U.S. equities market is comprised of more than five thousand stocks, so that, in principle, a verification of market efficiency in the U.S. equities market would require as many as five thousand separate tests. The following characterization of market efficiency simplifies the task considerably by reducing the number of tests to just two: NA and NUPBR. First, we introduce some helpful notation. Let

$$\alpha = (\alpha_0, \cdots, \alpha_{n-1})$$

be an n dimensional vector of real numbers such that $\alpha_i > 0$, and $\sum_{i=0}^{n-1} \alpha_i = 1$, so that α is a weight vector. Define $S^\alpha := \alpha \cdot S = \sum_{i=0}^{n-1} \alpha_i S^i$, i.e. S^α is the weighted sum of the n asset prices, and it is interpreted as the value process of the market portfolio computed using the weight vector α. Next, denote $\tilde{S}^\alpha$ the $n+1$ dimensional price vector obtained by appending S^α to S, i.e. $\tilde{S}^\alpha := (S, S^\alpha)$. Denote

$$Z^\alpha := \tilde{S}^\alpha (S^\alpha)^{-1}, \tag{2.5}$$

so that Z^α is a change of numeraire that restates S in units of the market portfolio. We now have the following:

2.4 Proposition *The market $(S, \mathcal{B})$ is efficient if and only if $(Z^\alpha, \mathcal{B})$ admits an equivalent local martingale measure for all strictly positive weight vectors α.*

Proof. Suppose, $(Z^\alpha, \mathcal{B})$ admits a local martingale measure while there exists a c-admissible strategy, $c > 0$,

$$H := (H^0, \cdots, H^{n-1})$$

for S and at least one $0 \leq k < n$ such that $(H \bullet S)_\infty \geq S^k_\infty - 1$ holds almost surely, with the inequality holding strictly with positive probability. Denote α_{-k} the vector obtained by substituting 0 for the k-th coordinate of α. Set $K := \alpha_{-k} + \alpha_k H$ and observe that $(K \bullet S)_\infty \geq S^\alpha_\infty - 1$, almost surely, with the inequality holding strictly on a set of positive measure. Set $J := (K, 0)$ and note that $J \bullet \tilde{S}^\alpha = K \bullet S$. Now consider

$$Y = (S^\alpha)^{-1}(J \bullet \tilde{S}^\alpha + 1) - 1.$$

Because $J_0 = 0$ and $S^\alpha_0 = 1$, we have $Y_0 = 0$. Because H is c-admissible and $0 < (S^\alpha_t)^{-1} \leq (\alpha_0)^{-1}$, we have $Y_t \geq (\alpha_0)^{-1}(1 - \alpha_0)(1 - c)$, and $P(Y_\infty \geq 0) = 1$ with $P(Y_\infty > 0) > 0$ because $H \bullet S$ dominates S^k. By the stochastic integration by parts formula, $Y = (J + I^{n+1,0}) \bullet Z^\alpha$. That is, Y is an arbitrage for Z^α.

Now suppose $(H^m)_{m \geq 1}$ violates NUPBR for S. Then there is $\beta > 0$ such that for all $N \in \mathbb{N}$, $P((H^m \bullet S)_\infty > N) > \beta$ for sufficiently large m. Let $Y^m = (S^\alpha)^{-1}(K^m \bullet \tilde{S}^\alpha + 1) - 1$, where $K^m = (H^m, 0)$. It is easy to see that $Y^m_0 = 0$, and $Y^m_t \geq -1$. Under the assumption of NFLVR, $(S^\alpha_\infty)^{-1}$ is well-defined, finite-valued, and contained in $(0, \alpha_0^{-1}]$ (Delbaen & Schachermayer, 1994, Theorem 3.3). Hence, there is a sufficiently small c such that $P((S^\alpha_\infty)^{-1} > c) > 1 - \beta/2$. Hence, for sufficiently large m,

$$P(Y^m_\infty > N) > P((K^m \bullet \tilde{S}^\alpha)_\infty > c^{-1}(N + 1) - 1) - P((S^\alpha_\infty)^{-1} \leq c),$$

which eventually exceeds $\beta/2$. Using Itô's integration by parts formula, it may be easily seen that Y^m is expressible as a stochastic integral with respect to Z^α.

Now suppose $(S, \mathcal{B})$ is efficient but for some α, $(Z^\alpha, \mathcal{B})$ admits an arbitrage. So that there is 1-admissible H such that $(H \bullet Z^\alpha)_\infty \geq 0$ almost surely with the inequality holding strictly with strict probability. Then it is easily verified, arguing as in the previous paragraphs, that $Y := (S^\alpha)(H \bullet Z^\alpha + 1) - 1$ is equal to $K \bullet S$ where K is 1-admissible for S. Because H is an arbitrage for Z^α, we have that $Y_\infty = (K \bullet S)_\infty \geq S^\alpha_\infty - 1$, with the inequality holding strictly with positive probability. Hence, S^α is dominated by K. That ND fails for at least one asset now follows from (Delbaen & Schachermayer, 1997, Proposition 2.12). Indeed, denote $J := \alpha_{n-1}^{-1}(K - \sum_{i=0}^{n-2} \alpha_i)$ and observe that $(J \bullet S)_\infty \geq S^{n-1}_\infty - 1$, almost surely, with the inequality holding strictly with positive probability. By the non negativity of S^{n-1}, we also have that $(J \bullet S)_\infty \geq -1$; by Proposition 2.11 of Delbaen & Schachermayer (1997), $(J \bullet S)_t \geq -1$ on $\mathbb{R}_+$, so that J is 1-admissible. This is a contradiction of the no dominance assumption on S^{n-1}.

Now, if $(H^m)_{m\geq 1}$ is a violation of NUPBR for Z^α then $(Y^m)_{m\geq 1}$, where $Y^m = (S^\alpha)(H^m \bullet Z^\alpha + 1) - 1$ violates NUPBR for S. □

As a corollary to the previous claim, we now have the following:

2.0.1 Corollary *Under the assumption of NUPBR for S, $(S, \mathcal{B})$ is efficient if and only if S^α is undominated for every strictly positive weight vector α.*

Proof. The necessity of the claim follows as in Proposition 2.4. On the other hand if S^i is dominated by H then $K \bullet S$, where $K = \alpha_{-i} + \alpha_i H$ and α_{-i} is the portfolio weight α with 0 substituted for α_i, dominates S^α. □

The next result shows that the choice of weight vector is irrelevant.

2.0.2 Corollary *Let α be a srictly positive weight vector. $(Z^\alpha, \mathcal{B})$ satisfies NFLVR if and only if $(Z^\kappa, \mathcal{B})$ satisfies NFLVR for all strictly positive weight vectors κ.*

Proof. Sufficiency is obvious. Suppose $(Z^\kappa, \mathcal{B})$ fails to satisfy NUPBR for a strictly positive weight vectors κ. Let $(H^m)_{m\geq 1}$ denote the sequence yielding unbounded profits in the market $(Z^\kappa, \mathcal{B})$. Then using familiar arguments, it is easily verified that $Y^m := (S^\kappa)(H^m \bullet Z^\kappa + 1) - 1$ constitutes a violation of NUPBR for $(S, \mathcal{B})$. By Proposition 2.4, $(Z^\alpha, \mathcal{B})$ cannot satisfy NFLVR.

Suppose $(Z^\kappa, \mathcal{B})$ satisfies NUPBR but fails to satisfy NA. Then arguing as in Proposition 2.4, it is easily seen that S^κ is dominated. By Corollary 2.0.1, $(S, \mathcal{B})$ cannot be efficient. By Proposition 2.4, $(Z^\alpha, \mathcal{B})$ cannot satisfy NFLVR. □

The next characterization of market efficiency is perhaps the most intuitive. It states that in a complete market setting, a market is efficient if and only if there exists Q equivalent to P such that all asset prices are *uniformly integrable* Q martingales. Hence, not only must risk-adjusted prices be unpredictable, they must also have constant unconditional risk-adjusted expectation across time and at infinity. This result is the infinite horizon counterpart of (Jarrow & Larsson, 2012, Theorem 3.2 (iii)).

2.5 Proposition *The market $(S, \mathcal{B})$ is efficient if and only if there exists an equivalent local martingale measure Q for S such that S is a uniformly integrable martingale under Q.*

Proof. The claim follows from (Delbaen & Schachermayer, 1995, Theorem 13). Indeed, by Proposition 2.4, efficiency is equivalent to $(Z^\alpha, \mathcal{B})$ admitting a local martingale measure. Let Q' be a local martingale measure for Z^α. Since $0 < (S^\alpha)^{-1} \leq \alpha_0^{-1}$ on $\mathbb{R}_+$, it follows in particular that $(S^\alpha)^{-1}$ is a uniformly integrable martingale under Q' (Protter, 2004, Theorem 51). Define $dQ = (S^\alpha_\infty)^{-1}dQ'$. That S^α is uniformly integrable follows from the fact that for all stopping times τ, $E^Q(S^\alpha_\tau) = 1$. Since $0 \leq \alpha_i S^i \leq S^\alpha$, we have that S^i is uniformly integrable as well. Moreover, $\tilde{S}^\alpha = Z^\alpha(S^\alpha)$ is a Q local martingale (He et al., 1992, Theorem 12.12). In particular, S is a Q local martingale. Hence, Q is an equivalent local martingale measure for S.

Suppose Q is a uniformly integrable martingale measure for S. Then Q doubles as a local martingale measure for S, so that NFLVR is satisfied (Delbaen & Schachermayer, 1999, Theorem1.1). It remains to show prices are not dominated. Suppose there is K admissible for S such that

$$P((K \bullet S)_\infty \geq S^i_\infty - 1) = 1, \tag{2.6}$$

with the inequality holding strictly with positive probability for some $0 \leq i < n$. Because S is a local martingale under Q, it follows that $K \bullet S$ is a σ-martingale, so that by (Ansel & Stricker, 1994, Corollary 3.5) it is a local martingale. Moreover, since it is bounded below, $K \bullet S$ is a supermartingale. Hence, under the assumption of uniformly integrable S^i, we have $E^Q((K \bullet S)_\infty - (S^i_\infty - 1)) = E^Q((K \bullet S)_\infty) \leq E^Q((K \bullet S)_0) = 0$. Since (2.6) holds and $P \sim Q$, it must be the case that $P((K \bullet S)_\infty = S^i_\infty - 1) = 1$. $\square$

According to Proposition 2.5, the equivalence holds only if there exists an equivalent probability measure Q such that prices, in addition to being martingales, are also *uniformly integrable*. Indeed, the following is a counterexample taken from (Delbaen & Schachermayer, 1999).

2.1 Example Let $(\varepsilon_m)_{m\geq 1}$ be independent and identically distributed Bernoulli sequence taking values 1 and -1 with equal probability (P). Let c denote a real number satisfying $0 < c < 1$ and define the price process $(S_m)_{m\geq 1}$ recursively as follows: $S_0 = 1$ and $S_m = c$ if $\varepsilon_m = 1$, and $S_m = 2S_{m-1} - c$ otherwise. Now consider an economy with two assets $(1, S)$. Denote $\mathcal{F}_m$ the σ-algebra generated by ε_m and observe that $E(S_m|\mathcal{F}_{m-1}) = 1/2(c + 2S_{m-1} - c) = S_{m-1}$. So that $(1, S)$ is a martingale for $(\mathcal{F}_m)_{m\geq 1}$ under P. Meanwhile, note that $S_\infty = c < 1$ almost surely since the probability of all occurrences of ε_m being -1 is zero. Hence, S is strongly dominated by 1. That is $(1, S)$ fails to satisfy ND and, therefore, market efficiency even though it is a martingale.

2.1 Statistical inference for market efficiency

In the asset management industry, an arbitrage is often understood, at least implicitly, as a trading strategy capable of generating positive expected excess return beyond the level implied by its exposure to a set of risk factors. The set of risk factors is often the return of a market index such as the S&P 500 together with the size and value factors of Fama & French (1993). This excess positive return beyond the level prescribed by the benchmark index or factors is often denoted α and the strategy as a whole is often referred to as an *alpha*. The economic appeal of an arbitrage is the possibility of achieving positive excess returns while incurring a less than commensurate amount of risk.

In other words, an arbitrage is a free lunch. Clearly, the free lunch interpretation of an arbitrage only makes sense to the extent that the benchmark factors accurately represent the sources of systematic risk present in the economy. As a case in point, a strategy based on the "small size effect" (Banz, 1981) produces positive alpha when systematic risk is proxied with the return on a market index; of course, the positive alpha vanishes in the multi-factor model of Fama & French. Hence, a true determination of an alpha, at least in the multi-factor framework, is only possible if the underlying risk factors are known and measurable with accuracy. Another, way to state the same thing is to consider the fact that in an exponentially affine multi-factor framework, the logarithm of the Radon-Nikodym derivative of the risk-neutral measure, is given by

$$m = a + \sum_{i=1}^{k} b_i f_i$$

where a and b_i are constants and f_i, $0 < i \leq k$, is a systematic/priced risk. Hence, a choice of $(f_i)_{0<i\leq k}$ may be viewed as expressing an opinion about m or indeed the risk-neutral measure Q since

$$Q(A) = \int_A \exp(m) dP \tag{2.7}$$

for all events A.

In practice, the pricing kernel m is unobservable so that the choice of risk factors $(f_i)_{0<i\leq k}$ is subject to error; indeed the choice of a linear relationship itself is subject to error. A means by which the misspecification error may be sidestepped is suggested by the *local* martingale characterization of market efficiency, i.e. a market is efficient in large financial markets if NA, ND, and NUPBR hold. These conditions are expressed in terms of the physical

measure, so that, by formulating empirical tests based on these concepts the misspecification error inherent in trying to estimate the pricing kernel m may be avoided.

By Proposition 2.4, if the aim is to study efficiency in the market $(S, \mathcal{B})$ then it may prove to be more efficient to first perform a change of numeraire, using the market portfolio as the new numeraire, and then studying the market $(Z^\alpha, \mathcal{B})$. This approach has the benefit of obviating the need to perform the ND test for each asset, since each violation of ND translates into a violation of NA for Z^α. To that end we have the following lemma.

2.6 Proposition *Let $(H^m)_{m\geq 1}$ be a sequence of admissible simple strategy for Z^α, that is H^m admits the representation $H^m = \sum_{i=1}^{n_m} \zeta_i I_{\rrbracket \tau_{i-1}, \tau_i \rrbracket}$, where $n_m \uparrow \infty$, τ_i is a stopping time, and ζ_i is $\mathcal{F}_{\tau_{i-1}}$ measurable. Further suppose that*

$$E^P((V^m_{\tau_m})^2) < \infty,$$

where $V^m_{\tau_m} := (H^m \bullet Z^\alpha)_{\tau_m}$. Suppose there is an admissible strategy H for Z^α such that $H^m \bullet S \xrightarrow{ucp} H \bullet S$. Then H constitutes a violation of NA for Z^α if and only if

$$\lim_m E^P(V^m_{\tau_m}) > \beta \text{ for some } \beta > 0, \tag{2.8}$$

$$\lim_m P(V^m_{\tau_m} < 0) = 0. \tag{2.9}$$

Moreover, if $(H^m)_{m\geq 1}$ denotes a sequence of 1-admissible simple strategies for Z^α such that

$$E^P(V^m_{\tau_m}) < \infty.$$

Then $(H^m)_{m\geq 1}$ constitutes a violation of NUPBR for Z^α if and only if

$$\lim_m E^P(V^m_{\tau_m}) = \infty. \tag{2.10}$$

Proof. These statements follow directly form the definitions of NA and NUPBR. □

The simplest way to verify (2.8), (2.9), and (2.10) is probably to specify a parametric model for the incremental payoffs of H^m. This is the approached taken in Jarrow et al. (2012) to study *statistical arbitrage* opportunities. Let $(\varepsilon_i)_{1\leq i\leq n_m}$ denote a sequence of independent standard normal variables and define

$$\Delta V^m_{\tau_i} := V^m_{\tau_i} - V^m_{\tau_{i-1}} = \mu i^\theta + \sigma i^\gamma \varepsilon_i, \tag{2.11}$$

where μ, θ, σ, and γ are constants. This specification is the unconstrained mean (UM) model of Hogan et al. (2004); this basic setup may be modified to accommodate more complicated behaviors such as correlated errors and coefficients that change from one small market to the next. Observe that $V^m_{\tau_m}$ is normally distributed with mean $\mu \sum_{i=1}^{n_m} i^\theta$ and variance $\sum_{i=1}^{n_m} (\sigma i^\gamma)^2$. The log likelihood function is given by

$$\mathcal{L}(\Theta) := -2^{-1} \sum_{i=1}^{n_m} \log(\sigma i^\gamma)^2 - (2\sigma^2)^{-1} \sum_{i=1}^{n_m} i^{-2\gamma} (\Delta V^m_{\tau_i} - \mu i^\theta)^2.$$

where $\Theta := (\mu, \theta, \sigma, \gamma)$. The parameter vector may be estimated in the usual manner by setting the gradient of $\mathcal{L}(\Theta)$ to zero and solving a system of four equations in four unknowns to obtain an estimate $\hat{\Theta} := (\hat{\mu}, \hat{\theta}, \hat{\sigma}, \hat{\gamma})$.

Now observe that if both μ and θ are positive then $(H^m)_{m \geq 1}$ constitutes a violation of the NUPBR condition for Z^α. If $\mu > 0$, $\gamma < 0$, and θ is sufficiently large then (H^m) converges to an arbitrage for Z^α and, by Proposition 2.4, a violation of market efficiency for $(S, \mathcal{B})$. In (Hogan et al., 2004, Theorem 6), it is shown that $\theta > \gamma - 1/2 \vee -1$ is sufficient to ensure convergence to an arbitrage. The above considerations are summarized in the following Proposition.

2.7 Proposition *Under the assumptions of Lemma 4.1, if the incremental payoffs of H^m satisfy* (2.11) *then the null hypothesis of market efficiency may be rejected with $1 - \alpha$ confidence if either one of the joint tests*

1. $H_1 : \hat{\mu} > 0$ *and* $H_2 : \hat{\theta} > 0$, *or*
2. $H_1' : \hat{\gamma} < 0$, $H_2' : \hat{\mu} > 0$, $H_3' : \hat{\theta} > \hat{\gamma} - 1/2 \vee -1$.

achieve a combined p-value of less than α.

It is worth noting that since these tests involve the specification of a model for the incremental payoffs of the target strategies, they are subject to misspecification errors. Hence, these tests also involve testing a joint-hypothesis. The advantage of the current tests over traditional tests that require the specification of a model for the stochastic discount factor is that, the misspecification error incurred in the tests proposed here may be analyzed and tested; this is so because they only require observable (at least at discrete times) data: prices and portfolio returns. This is in contrasts to the "unmeasurable" misspecification error incurred in traditional tests which rely on estimates of unobservable quantities such as the stochastic discount factor underlying the market.

Moreover, the incremental payoff specification in (2.11) is just one example. Another reasonable model that may be analyzed by maximum likelihood methods would involve modeling incremental payoffs as the sum of an exponential random variable and a Gaussian random variable. The positivity of the volatility of the Gaussian component can then be tested as m tends to infinity to verify violations of the no arbitrage condition. Clearly, the model that is ultimately selected would depend on how well it fits the data being analyzed.

3 Market efficiency in large financial markets

The theory of large financial markets is a modern re-imagining of the arbitrage pricing theory (APT). The APT (Ross, 1976a,b) was devised as an alternative to the capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965); it aims to obviate the need for accurate measures of the market portfolio and to relax some of the assumptions underlying the CAPM. It assumes that changes in individual asset returns are due to changes in a fixed number of factors plus an uncorrelated idiosyncratic component. Under the assumption of *no arbitrage* (Huberman, 1982), the security-market line is approximated arbitrarily well, as the number of assets increases without bound.

The APT is fundamentally a discrete time theory. The theory of *large financial markets* was introduced in (Kabanov & Kramkov, 1994, 1998) as a dynamic continuous-trading extension of the APT. In this modern incarnation, the APT employs the tools of mathematical finance pioneered by Harrison & Pliska (1981). A large financial market is defined as a sequence of *small* markets $(S^n, \mathcal{B}^n, T^n)$, $n \in \mathbb{N}$, where $0 < T^n \leq \infty$ is the terminal time in the n-th small market, S^n is a d_n-dimensional vector of asset prices and $\mathcal{B}^n$ is a filtered probability basis $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, P^n)$. In the sequel, we adopt the *large financial market on one probability space* setting of Cuchiero et al. (2015) with $d_n = n$, $n \in \mathbb{N}$, i.e. $T^n = T < \infty$, $\mathcal{B}^n = \mathcal{B}$, $n \in \mathbb{N}$, and $(S^n)_{n\geq 1}$ forms a nested sequence of n-dimensional asset prices, so that the i-th price process in S^n is indistinguishable from the i-th coordinate of S^m whenever $0 \leq i \leq n \leq m$.

In the classic small market setup treated in the previous section, market efficiency is characterized in terms of NFLVR and the no dominance condition. We introduce a similarly motivated definition of market efficiency in large financial markets in terms of asymptotic no free lunch with vanishing risk (ANFLVR), the large financial market counterpart of NFLVR (Cuchiero et al., 2015), and asymptotic no dominance (AND) defined below (Definition

4.1). We begin with the introduction of large financial market notation and definitions.

3.1 Large financial market payoff space

We adopt the notation of Cuchiero et al. (2015). Given the n-th small market $(S^n, \mathcal{B})$, where S^n is an n-dimensional semimartingale representing asset prices, a λ-admissible strategy, $\lambda > 0$, is a predictable process H such that $H_0 = 0$, the stochastic integral $H \bullet S^n$ is well-defined, and $H \bullet S^n_t \geq -\lambda$ for all $0 \leq t \leq T$. We will call $X^n := H \bullet S^n$ an admissible gain process if H is λ-admissible for some positive real λ. We will denote by $\mathcal{X}^n_\lambda$ the set of λ-admissible gain processes and by $\mathcal{X}^n$ the collection of all admissible processes in $(S^n, \mathcal{B})$, i.e.

$$\mathcal{X}^n := \bigcup_{\lambda > 0} \mathcal{X}^n_\lambda = \bigcup_{\lambda > 0} \lambda \mathcal{X}^n_1.$$

Small market payoff spaces are denoted $\mathcal{K}^n$ and $\mathcal{K}^n_1$ and defined as the terminal values of small market gain processes:

$$\mathcal{K}^n := \{X_T : X \in \mathcal{X}^n\}, \text{ and } \mathcal{K}^n_1 := \{X_T : X \in \mathcal{X}^n_1\}.$$

The space of small market dominated payoffs are defined in the classical manner:

$$\begin{aligned} \mathcal{C}^n_0 &:= \{f - g : f \in \mathcal{K}^n \text{ and } g \in L^0_+(\mathcal{B})\}, \\ \mathcal{C}^n &:= \{f : f \in \mathcal{C}^n_0 \text{ and } f \in L^\infty(\mathcal{B})\}. \end{aligned}$$

Now for an adapted càdlàg process X carried on the basis $\mathcal{B}$, denote $(X)^*_T := \sup_{s \leq T} |X_s|$ and define

$$\|X\|_{ucp} = E(\min((X)^*_T, 1)).$$

The functional $\| \cdot \|_{ucp}$ is a quasi-norm, and it induces a complete metric $d_{ucp}(X, Y) := \|X - Y\|_{ucp}$ on the space of adapted càdlàg processes. We employ the notation $X^n \xrightarrow{ucp} X$ to denote convergence with respect to this topology. A predictable process H will be called *simple* if there exists $\mathbb{F}$-stopping times $0 = S_0, \cdots, S_{k+1} = T$, and $\xi_i \in \mathcal{F}_{S_i}$ with $\|\xi_i\|_\infty < \infty$, $0 \leq i \leq k$, such that

$$H_t = \xi_0 I_{[\![0]\!]}(t) + \sum_{i=1}^{k} \xi_i 1_{]\!]S_i, S_{i+1}]\!]}(t).$$

In the sequel, ξ_0 is assumed to be identically zero. We denote by Λ the set of $\mathcal{B}$-predictable simple processes. Next, for a càdlàg adapted process X, define

$$\|X\|_{\mathcal{S}} := \sup\{\|H \cdot X\|_{ucp} : H \in \Lambda, |H| \leq 1\}.$$

The functional $\|\cdot\|_{\mathcal{S}}$ induces a complete metric space on the space of semimartingales referred to interchangeably as the Emery or semimartingale topology. We employ the notation $X^n \xrightarrow{\mathcal{S}} X$ to denote convergence with respect to this topology. Now, a process X is said to be a 1-admissible generalized gain process if there exists a sequence of small market wealth portfolios $X^n \in \mathcal{X}_1^n$ such that

$$X^n \xrightarrow{\mathcal{S}} X,$$

that is, X is a limit point in the semimartingale topology of $\bigcup_{n\geq 1} \mathcal{X}_1^n$. We denote the set of λ-admissible generalized wealth portfolios by $\mathcal{X}_\lambda$ and the set consisting of all admissible generalized wealth portfolios by $\mathcal{X}$, i.e.

$$\mathcal{X} := \bigcup_{\lambda>0} \mathcal{X}_\lambda = \bigcup_{\lambda>0} \lambda\mathcal{X}_1.$$

We now define the payoff spaces $\mathcal{K}$ and $\mathcal{K}_1$ as the terminal values of generalized wealth portfolios:

$$\mathcal{K} := \{X_T : X \in \mathcal{X}\}, \text{ and } \mathcal{K}_1 := \{X_T : X \in \mathcal{X}_1\}.$$

Given the above, we define the set of large financial market dominated payoffs as follows:

$$\mathcal{C}_0 := \{f - g : f \in \mathcal{K} \text{ and } g \in L^0_+(\mathcal{B})\},$$
$$\mathcal{C} := \{f : f \in \mathcal{C}_0 \text{ and } f \in L^\infty(\mathcal{B})\}.$$

3.2 Arbitrage pricing in large financial markets

We now recall the fundamental theorem of asset pricing for large financial markets (Cuchiero et al., 2015, Theorem 1.1). That is, necessary and sufficient conditions with acceptable economic interpretations under which the existence of a pricing functional (equivalent separating measure) is assured. Since zero is contained in $\mathcal{C}$, we would like the pricing functional or more specifically the P-equivalent probability measure Q to satisfy $E^Q(f) \leq 0$ for all $f \in \mathcal{C}$. In order to make these statements precise in the large financial market setting, we require the following definitions and lemmas.

3.1 Lemma *If $f \in \mathcal{C}_0$ then there exists $f_n \in \mathcal{C}_0^n$, $n \in \mathbb{N}$, such that $f_n \xrightarrow{P} f$.*

Proof. Suppose $f \in \mathcal{C}_0$. Then there is $X \in \mathcal{X}$ and random variable $g \in L^0_+(\mathcal{B})$ such that $f = X_T - g$. Since $X \in \mathcal{X}$, there is $X^n \in \mathcal{X}^n$ such that $\|X^n - X\|_{\mathcal{S}} \to 0$. This in turn implies that $X^n \xrightarrow{ucp} X$, so that $X^n_T \xrightarrow{P} X_T$. Set $f_n := X^n_T - g$. Then $f_n \in \mathcal{C}_0^n$, and $f_n \xrightarrow{P} f$. □

Hence, the dominated payoff of a generalized gain process may be viewed as the limit of dominated payoffs in small markets. The next lemma shows that the same can be said for the bounded portion of $\mathcal{C}_0$.

3.2 Lemma *If $f \in \mathcal{C}$ then there exists $f_n \in \mathcal{C}^n$ such that $f_n \xrightarrow{P} f$.*

Proof. Let $f \in \mathcal{C}$, then $f \in \mathcal{C}_0$ and $f \in L^\infty(\mathcal{B})$, i.e. there exists a $K < \infty$ such that $f \leq K$ almost surely. Because $f \in \mathcal{C}_0$, there exists, by Lemma (3.1), $g_n \in \mathcal{C}_0^n$ such that $g_n \xrightarrow{P} f$. Set $f_n := g_n - (g_n - K)I_{\{g_n \geq K\}}$. Then $f_n \in \mathcal{C}^n$, and $f_n \xrightarrow{P} f$. □

3.1 Definition *A large financial market $(S^n, \mathcal{B})_{n\geq 1}$ is said to possess the (Asymptotic) No Arbitrage (ANA) property if there does not exist $X^n \in \mathcal{X}_1^n$, $n \in \mathbb{N}$, and $X \in \mathcal{X}_1$ such that $\|X^n - X\|_{\mathcal{S}} \to 0$ and*

$$\limsup_n P(X^n_T < 0) = 0, \tag{3.12}$$

$$\liminf_n P(X^n_T > \alpha) > \alpha, \tag{3.13}$$

for some $\alpha > 0$.

It is easily verified that the definition of ANA given here is equivalent to the more familiar functional analysis definition:

$$\mathcal{K}_1 \cap L^0_+(\Omega, \mathcal{F}, P) = \{0\}.$$

Because our interests are econometrically motivated, Definition 3.1 is more natural. The next definition is the large market counterpart of NUPBR.

3.2 Definition *A large financial market $(S^n, \mathcal{B})_{n\geq 1}$ is said to satisfy the No Unbounded Profit with Bounded Risk (NUPBR) condition if $\mathcal{K}_1$ is bounded in $L^0(\mathcal{B})$.*

These two notions of arbitrage are equivalent to our next notion of arbitrage (Cuchiero et al., 2015, Proposition 4.4).

3.3 Definition *A large financial market* $(S^n, \mathcal{B})_{n\geq 1}$ *is said to possess the Asymptotic No Free Lunch with Vanishing Risk (ANFLVR) property if*

$$\overline{\mathcal{C}} \cap L^\infty_+(\Omega, \mathcal{F}, P) = \{0\},$$

where $L^\infty_+(\Omega, \mathcal{F}, P)$ *denotes the set of essentially bounded nonnegative random variables on* $\mathcal{B}$ *and* $\overline{\mathcal{C}}$ *is the norm closure of* $\mathcal{C}$ *in* $L^\infty(\Omega, \mathcal{F}, P)$*.*

It is shown in (Cuchiero et al., 2015, Theorem 1.1) that a version of the fundamental theorem of asset pricing holds in the large financial market setting: ANFLVR is necessary and sufficient for the existence of an equivalent separating measure (ESM), where an ESM is a probability Q equivalent to P such that $E^Q(f) \leq 0$ for $f \in \mathcal{C}$.

4 Asymptotic Market efficiency

In the standard small market setting, the simultaneous satisfaction of the NFLVR condition and the ND property for all assets is equivalent to market efficiency. In the case of non-negative asset prices, it is also the case that there exists a Q equivalent to P such that prices are uniformly integrable martingales (Proposition 2.5). Here, our objective is to extend these notions to the framework of large financial markets. We start with an adaptation of the ND condition to the large financial market setting. For each n we assume that S^n is an n-dimensional semimartingale with the zeroth component $S^{n,0} = 1$ on $[0, T]$. So that, we have $d_n = n$. For all $0 \leq i < n$, the i-th asset price satisfies $S^{n,i}_t \geq 0$ for $t \in [0, T]$. We also assume time zero prices are deterministic and that the entire price process is normalized so that $S^{n,i}_0 = 1$ for $0 \leq i < n \in \mathbb{N}$.

4.1 Definition (Asymptotic No Dominance (AND)) *A large financial market payoff* $f \in \mathcal{K}$ *is said to be (asymptotically) undominated if for all* $g \in \mathcal{K}$ *if* $g \geq f$*, a.s., then it must also be the case that* $g = f$ *almost surely.*

Now let $A^k := \{a_0, a_1, \cdots, a_{k-1}\}$ and denote an arbitrary set of $k > 0$ distinct natural numbers including 0; we adopt the convention

$$a_0 = 0.$$

Now let

$$\alpha_k := (\alpha_{a_0}, \alpha_{a_1}, \cdots, \alpha_{a_{k-1}})$$

denote a strictly positive weight vector, that is $\sum_{j=0}^{k-1} \alpha_{a_j} = 1$, and $\alpha_{a_j} > 0$ for $0 \leq j < k$. Now, for $n \geq \max\{a : a \in A^k\}$ define

$$S^{\alpha_k} = \sum_{j=0}^{k-1} \alpha_{a_j} S^{n,a_j}.$$

We will refer to S^{α_k} as the *convex portfolio* generated by (A^k, α_k). Note that because $(S^n)_{n\geq 1}$ is a nested sequence, S^{α_k} is up to an evanescent set independent of n for $n \geq \max\{a : a \in A^k\}$.

4.2 Definition (Asymptotic Market Efficiency (AME)) *A large financial market $(S^n, \mathcal{B})_{n\geq 1}$ is said to be asymptotically efficient on $[0, T]$ if*

1. *ANFLVR holds for $(S^n, \mathcal{B})_{n\geq 1}$, and*
2. *for all convex portfolios S^{α_k}, the payoff $S_T^{\alpha_k} - 1$ is asymptotically undominated.*

Now denote S^{n,α_k} the $n + 1$ dimensional vector obtained by appending S^{α_k} to S^n, that is $S^{n,\alpha_k} = (S^n, S^{\alpha_k})$. Define

$$Z^{n,\alpha_k} = S^{n,\alpha_k}(S^{\alpha_k})^{-1}.$$

Hence, Z^{n,α_k} expresses S^n in units of S^{α_k}.

4.1 Proposition *The large financial market $(S^n, \mathcal{B})_{n\geq 1}$ satisfies NUPBR if and only if $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$ with $n_k = \max\{a : a \in A^k\}$ satisfies NUPBR for all Z^{n,α_k}.*

Proof. Suppose $(H^n)_{n\geq 1}$ violates NUPBR for $(S^n, \mathcal{B})_{n\geq 1}$. Then there exists $\beta > 0$ such that for $N \in \mathbb{N}$ and sufficiently large n, we have $P((H^n \bullet S^n)_T > N) > \beta$. Consider

$$Y^n := (S^{\alpha_k})^{-1}(H^n \bullet S^n + 1) - 1.$$

Because all prices have initial value 1 and $H_0^n = 0$, we have $Y_0^n = 0$. Because $(S^{\alpha_k})_T^{-1}$ is finite-valued, $(Y_T^n)_{n\geq 1}$ is unbounded in $L^0(\mathcal{B})$. Because H^n is 1-admissible, $Y^n \geq -1$ on $[0, T]$. By Itô's integration by parts formula and the fact that $Z^{n,\alpha_k} = S^{n,\alpha_k}(S^{\alpha_k})^{-1}$, we have $Y^n = K^n \bullet Z^{n,\alpha_k}$ for a predictable K^n. Hence, $(K^n)_{n\geq n_k}$ violates NUPBR for $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$.

For the converse denote $(H^n)_{n\geq n_k}$ a violation of NUPBR for $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$ and consider

$$Y^n := (S^{\alpha_k})(H^n \bullet Z^{n,\alpha_k} + 1) - 1.$$

That $(Y^n)_{n\geq 1}$, with $Y^n = 0$ for $n < n_k$, constitutes a violation of NUPBR for $(S^n, \mathcal{B})_{n\geq 1}$ follows by repeating the arguments of the previous paragraph. □

4.2 Proposition *Suppose $n \geq \max\{a : a \in A^k\} =: n_k$. Then S^{α_k} is asymptotically undominated if and only if $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$ satisfies ANA.*

Proof. Suppose S^{α_k} is dominated by $X \in \mathcal{X}$. Since $X \in \mathcal{X}$, there is $\lambda > 0$ and $X^n \in \mathcal{X}^n_\lambda$, $n \geq n_k$, such that $\|X^n - X\|_{\mathcal{S}} \to 0$. Because $X^n \in \mathcal{X}^n_\lambda$, we have $X^n = H^n \bullet S^n$ for H^n that is λ-admissible for S^n. Define $J^n := (H^n, 0)$ and observe that $J^n \bullet S^{n,\alpha_k} = H^n \bullet S^n$. Consider

$$Y^n := (S^{\alpha_k})^{-1}(J^n \bullet S^{n,\alpha_k} + 1) - 1. \tag{4.14}$$

Then $Y^n_0 = 0$, and $Y^n_t \geq (\alpha_{a_0})^{-1}(1 - \lambda) - 1$ for t in $[0, T]$. By Itô's integration by parts formula and the fact that $Z^{n,\alpha_k} = (S^{\alpha_k})^{-1} S^{n,\alpha_k}$, there is a predictable G^n such that $Y^n = G^n \bullet Z^{n,\alpha_k}$. By the foregoing, G^n is admissible for $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq 1}$. Because of the stability of convergence in the Emery topology (Kardaras, 2013, Proposition 2.10), we have $Y^n = G^n \bullet Z^{n,\alpha_k} \xrightarrow{\mathcal{S}} (S^{\alpha_k})^{-1}(W + 1) - 1 =: Y$. Hence, Y is a generalized gain process for S^{α_k}. Because W dominates S^{α_k}, we see that Y is an arbitrage for $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$.

Now suppose $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq 1}$ fails to satisfy arbitrage, so that the constant 1 is dominated by X_T where X is a 1-admissible generalized gain process for $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$. Then there is $(H^n)_{n \geq n_k}$ such that $H^n \bullet Z^{n,\alpha_k} =: X^n$ is a 1-admissible gain process for Z^{n,α_k}, and $X^n \xrightarrow{\mathcal{S}} X$. Consider

$$Y^n := S^{\alpha_k}(H^n \bullet Z^{n,\alpha_k} + 1) - 1.$$

Then $Y^n_0 = 0$, and $Y^n \geq -1$ on $[0, T]$. By Itô's integration by parts formula, there is predictable K^n such that $Y^n = K^n \bullet S^n$ is well-defined. We have by (Kardaras, 2013, Proposition 2.10) that $Y^n \xrightarrow{\mathcal{S}} S^{\alpha_k}(X + 1) - 1 =: Y$. Since X_T is a nonnegative and strictly positive with positive probability, we have that Y_T dominates $S^{\alpha_k}_T - 1$. □

4.1 Theorem *The large financial market $(S^n, \mathcal{B})_{n \geq 1}$ is asymptotically efficient if and only if $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$ satisfies ANFLVR for all (A^k, α_k).*

Proof. This follows from Proposition 4.1, 4.2, and (Cuchiero et al., 2015, Proposition 4.4). □

4.1 Statistical inference for asymptotic market efficiency

The small market tests discussed in the previous section hold under the assumption that the time horizon tends to infinity while the number of assets remains fixed. We perhaps draw an analogy with the *time series* regression tests of discrete-time empirical asset pricing (Cochrane, 2001, Chapter 12). In the current large financial market setup, the time horizon is held fixed while the number of assets is allowed to grow without bound. The empirical tests we propose in this section may be analogized to the *cross-section* regression tests of discrete-time empirical asset pricing theory.

Because the time horizon is assumed fixed, these tests may be particularly well-suited for analyzing strategies with short investment horizons. Also, since the cross-section is assumed to grow without bound, they may be more appropriate for studying strategies involving a great number of asset. In particular, strategies that involve sorting a great number of asset according some indicator of performance such as previous-year return. Examples of such strategies include mean-reversion and momentum strategies.

4.1 Lemma *Let (A_k, α_k) be given and let $(H^n)_{n\geq n_k}$ be a sequence of small market strategies for $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$ converging in the semimartingale topology to a generalized gain process Y. Suppose*

$$E((V_T^n)^2) < \infty,$$

where $V_T^n := (H^n \bullet Z^{n,\alpha_k})_T$. Then Y_T constitutes a violation of ANA for $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$ if and only if

$$\lim_n E(V_T^n) > \beta \text{ for some } \beta > 0, \tag{4.15}$$

$$\lim_n P(V_T^n < 0) = 0. \tag{4.16}$$

Moreover, if $(H^n)_{n,n_k}$ is a sequence of 1-admissible strategies for Z^{n,α_k} such that

$$E(V_T^n) < \infty$$

then $(H^n \bullet Z^{n,\alpha_k})_{n\geq n_k}$ violates NUPBR for $(Z^{n,\alpha_k}, \mathcal{B})_{n\geq n_k}$ if and only if

$$\lim_n E^n(V_T^n) = \infty. \tag{4.17}$$

Proof. These statements follow directly form the definitions of ANA and NUPBR.

The simplest way to determine whether a given strategy verifies the requirements of either (4.15), (4.16), or (4.17), is to specify a parametric model of its incremental payoffs. As a simple example, we may suppose that

$$\Delta V_T^i := V_T^i - V_T^{i-1} = \mu i^\theta + \sigma i^\gamma \varepsilon_i, \tag{4.18}$$

where μ, θ, σ, and γ are constants, (ε_i) is i.i.d, and ε_i is a standard normal random variable for $i \geq n_k$.

Now note that under the assumption of normally distributed ε_i, V_T^n is normally distributed with log likelihood given by

$$\mathcal{L}(\Theta) := -2^{-1} \sum_{i=1}^{n} \log(\sigma i^{\gamma})^2 - (2\sigma^2)^{-1} \sum_{i=1}^{n} i^{-2\gamma} (\Delta V_T^i - \mu i^{\theta})^2.$$

where $\Theta := (\mu, \theta, \sigma, \gamma)$. The parameter vector may be estimated in the usual fashion by setting the gradient of $\mathcal{L}(\Theta)$ to zero and solving a system of four equations in four unknowns to obtain an estimate $\hat{\Theta} := (\hat{\mu}, \hat{\theta}, \hat{\sigma}, \hat{\gamma})$. We summarize the preceding considerations as follows:

Now observe that if both μ and θ are positive then $(H^n)_{n \geq n_k}$ constitutes a violation of the NUPBR condition for $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$. If $\mu > 0$, $\gamma < 0$, and θ is sufficiently large then $(H^n \bullet Z^{n,\alpha_k})_{n \geq n_k}$ converges to an arbitrage for $(Z^{n,\alpha_k}, \mathcal{B})_{n \geq n_k}$ and, by Theorem 4.1, a violation of market efficiency for $(S^n, \mathcal{B})_{n \geq 1}$. In (Hogan et al., 2004, Theorem 6), it is shown that $\theta > \gamma - 1/2 \vee -1$ is necessary to ensure convergence to an asymptotic arbitrage. The above considerations are summarized in the following Proposition.

4.3 Proposition *Under the assumptions of Lemma 4.1, if the incremental payoffs of H^m satisfy* (4.18) *then the null hypothesis of market efficiency may be rejected with $1 - \alpha$ confidence if either one of the joint tests*

1. $H_1 : \hat{\mu} > 0$ *and* $H_2 : \hat{\theta} > 0$, *or*
2. $H_1' : \hat{\gamma} < 0$, $H_2' : \hat{\mu} > 0$, $H_3' : \hat{\theta} > \hat{\gamma} - 1/2 \vee -1$.

achieve a combined p-value of less than α.

5 Conclusion

In a finite horizon complete market setting, market efficiency is equivalent to asset prices admitting a martingale measure. This basic definition motivates traditional tests of market efficiency. These tests must by necessity postulate an equilibrium model of asset prices or a stochastic discount factor as a reference. Naturally, such a procedure is subject to a misspecification which cannot be assessed due to the fact that the stochastic discount factor (SDF) is unobservable. Hence, traditional tests of market efficiency are in fact joint tests of the fit of the particular model selected and deviations from market efficiency. This is the well-known joint hypothesis problem.

We have contributed to the growing literature that aims to devise tests of market efficiency that do not suffer from the joint-hypothesis problem. We have obtained further characterizations of market efficiency that in turn suggest simplifications of empirical tests of market efficiency. These characterizations involve a change of numeraire that boils down to normalizing asset prices with respect to the market portfolio prior to investigating violations of market efficiency. Our analysis may be extended to the large financial market setting. We define the no dominance condition as well as market efficiency in the large financial market framework. We show that the no dominance condition can be characterized in terms of the no arbitrage condition after a change of numeraire. This result suggest empirical tests of asymptotic market efficiency similar to those proposed in the small market setting. The practical importance of the large financial market theory is that for certain strategies, taking limits as the time horizon tends to infinity may be inappropriate. Provided the number of assets involved in the execution of the strategy is very large then the large financial market tests we proposed may be more adequate.

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