School of Finance

University of St.Gallen

STATE SPACE GEOMETRY OF ASSET PRICING: AN INTRODUCTION

IGOR POZDEEV

WORKING PAPERS ON FINANCE NO. 2015/18

SWISS INSTITUTE OF BANKING AND FINANCE (S/BF – HSG)

AUGUST 2015
THIS VERSION: MARCH 2016

State Space Geometry of Asset Pricing: An Introduction

Igor Pozdeev*

March 8, 2016

First draft: September 15, 2015

Abstract

This note is aimed at familiarizing the reader with state space geometry, a useful tool in teaching asset pricing concepts. Building on the analogy between expectation and dot product, I visualize basic notions using Euclidean geometry in 2D and 3D. Numerical examples are given such that the reader could easily follow the explanations.

*Swiss Institute of Banking and Finance, University of St.Gallen. I would like to thank Prof. Matthias Fengler at the University of St.Gallen for his support and contribution to this writing.

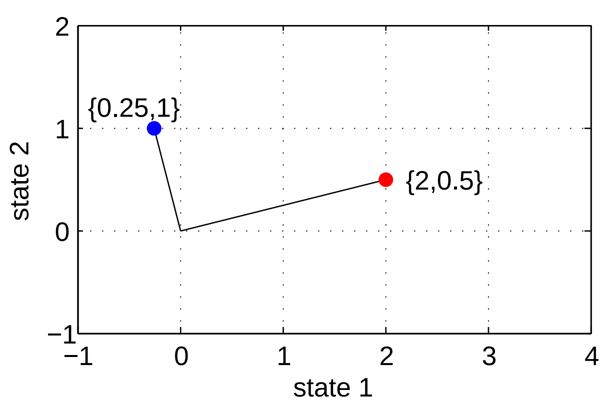
1 Introduction

State space geometry is a remarkably handy way of looking at things like returns, payoffs and prices. I have structured this tutorial the way I was learning the state space geometry some time ago which means it will be a mash-up of linear algebra, geometry and finance. As such, it might not be the best reference when it comes to any of these three fields in particular, but a decent one when we talk about their intersection.

I will explain most of the concepts like inner products and projections. The first part will deal with a two-asset economy, leaving the three-asset one for later. In what follows every time you see a row vector it will mean payoffs of *one* particular asset in different states; column vectors will thus stand for a cross-section of different assets' payoffs in one particular state. All vectors are meant to be column vectors unless explicitly written with a transpose sign. To denote the payoff of asset i in state s I will always write $x_i(s)$.

It is possible to plot payoffs on a set of S axes (S is the number of states) as dots with x coordinate being payoff in state one, y coordinate being payoff in state two etc. Figure 1 below depicts a simple state space with two states and two assets: the red asset pays 2 units in state one and 1 unit in state two; the blue asset pays -0.25 in state one and 1 in state two. One can think of these assets as of vectors, and I have indeed drawn black lines connecting the dots to the origin in order to make it more apparent. The major difference though is that these are random vectors, since their elements are possible realizations of some random variables with probabilities corresponding to probabilities of states. To avoid confusion in many analogies I will denote non-random vectors with an arrow above, e.g. $\vec{u}$.

Figure 1: **Two-asset economy.** A simple state space.



In asset pricing one often uses the expectation dot product, which is the joint second noncentral moment:

$$\mathbb{E}[uv] = u(1)v(1)\pi(1) + u(2)v(2)\pi(2) + \ldots + u(S)v(S)\pi(S) \tag{1}$$

where $\pi(s)$ is the probability that state s occurs. This is the counterpart of the "usual" dot product of two non-random vectors:

$$\vec{u} \cdot \vec{v} = \vec{u}(1)\vec{v}(1) + \vec{u}(2)\vec{v}(2) + \ldots + \vec{u}(S)\vec{v}(S) \tag{2}$$

If the dot product of two non-random vectors is zero they are said to be perpendicular to each other and make an angle of 90° when plotted, like the blue and the red payoffs in Figure 1. If the joint second noncentral moment of two random vectors is zero, they are said to be orthogonal, but unfortunately will not be perpendicular on a graph in the state space except when all states are equally likely to occur. To ease the understanding of concepts involving orthogonality of two vectors, I will always plot vectors as perpendicular if they are orthogonal in the stochastic sense. Don't worry: all the calculated quantities in this text will be correct but it will be not these values that I will use to draw pictures.

2 2D

2.1 Setting

Suppose we have two states of the world and two assets (the market is complete) with payoffs:

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{3}$$

This is to be read as: the first asset pays 1 unit of currency both in state one and two, the second asset pays nothing in state one and 2 in state two. Probabilities of the states are the following row vector:

$$\pi = \begin{bmatrix} 2/5 & 3/5 \end{bmatrix} \tag{4}$$

And prices of the two assets are the column vector below:

$$P = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \tag{5}$$

I calculate returns by dividing each row of X through the corresponding asset's price:

$$R = \begin{bmatrix} 1.5 & 1.5 \\ 0 & 6 \end{bmatrix} \tag{6}$$

and form excess returns by subtracting returns of the first asset from the rows of R:

$$R^e = \begin{bmatrix} 0 & 0 \\ -1.5 & 4 \end{bmatrix} \tag{7}$$

Stochastic discount factor (SDF) is calculated as:

$$M = P'\mathbb{E}[XX']^{-1}X = \begin{bmatrix} 5/4 & 5/18 \end{bmatrix} \tag{8}$$

Prices of Arrow-Debreau securities (henceforth AD prices, *a.k.a.* state prices) are given by:

$$Q = (X^{-1}P)^{\top} = \begin{bmatrix} 1/2 & 1/6 \end{bmatrix} \tag{9}$$

And the price of the risk-free security (equal to the expected value of SDF) is the sum of AD prices:

$$\mathbb{E}[M] = 1/R_f = 2/3 \tag{10}$$

which is of course the price of the first asset. Risk-neutral probabilities are (in complete markets) just AD prices divided through the sum of AD prices:

$$\pi^* = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix} \tag{11}$$

State one is in our setting the low-consumption state: that is why investors have priced the first asset, which pays nothing in state 1, so low, and that is why the risk-neutral probability of this state is so high.

2.2 P=0

Now, let us draw a 2D space with payoffs in state one on the x-axis and those in state two on the y-axis. The line of excess returns hosts assets with $P = 0$ and goes through the excess returns of the second asset as well as through $\{0, 0\}$, which allows us to write down the line equation:

$$\begin{cases} R_1^e(2) & = a + bR_1^e(1) \\ 0 & = a + b \times 0 \end{cases} \tag{12}$$

Note that the $P = 0$ line *always* goes through the origin since this is the necessary and sufficient condition for the **law of one price** to hold. Zero payoffs must have zero price.

Let us calculate the slope of this line in a different way: we know that the price of any asset is equal to the sum of its payoffs in each state times the corresponding AD price:

$$P(x) = x(1)q(1) + x(2)q(2) \tag{13}$$

For excess returns which have zero price this translates into:

$$0 = R^e(1)q(1) + R^e(2)q(2) \tag{14}$$

$$R^e(2) = -\frac{q(1)}{q(2)}R^e(1) \tag{15}$$

Hence the slope of the line is equal to the negative of the ratio of state prices. I plot the $P = 0$ line together with the excess return of the second asset in Figure 2.

I calculate the slope to be -3.0. This means that for each additional dollar of consumption in state one investors are ready to give up 3 dollars from their budget in state two. For angles less (more negative) than $-45°$ state 1 is the "hungry" state and vice versa.

2.3 P=1

Now let us draw the line of returns, or payoffs with price equal to 1. It must be parallel to the $P = 0$ line, otherwise the two would cross somewhere establishing a case when two assets with identical payoffs have different prices: a direct violation of the law of one price. We therefore don't have to calculate the slope again, and the intercept will be the payoff in state two of an asset with price 1 and zero payoff in state one:

$$1 = x(2)q(2) \tag{16}$$

$$x(2) = \frac{1}{q(2)} \tag{17}$$

The red line in Figure 3 is the $P = 1$ line.

We can go on and construct lines of payoffs with any price. The dashed line in Figure 3 shows the payoff $\{1, 1\}$ (black dot): its price is the price of a risk-free security, and we see that this is the same price as the one which would buy you 4 dollars of consumption in state two and nothing in state one.

2.4 Projections, R^* and R^{e*}

I move on to the two special payoffs: R^* and R^{e*}. The former is the payoff which can serve as SDF divided through its price:

$$R^* = \frac{x^*}{\mathbb{E}[x^{*2}]} \tag{18}$$

Taking M as a possible SDF and calculating its price with Arrow-Debreau securities:

$$R^* = \frac{M}{QM'} = \left[\begin{array}{cc} 54/29 & 12/29 \end{array}\right] \tag{19}$$

The second payoff is a projection of the risk-free vector $\{1, 1\}$ onto the space (in our case, the line) of excess returns:

$$R^{e*} = \text{proj}(1|R^e) \tag{20}$$

If you have two vectors $\vec{u}$ and $\vec{v}$ you can think of $\vec{u}$ as a hypotenuse of the triangle with one of the catheti being parallel and the other one perpendicular to $\vec{v}$ (see Figure 4). Projection is the parallel component, the "shadow" of $\vec{u}$ if the light falls perpendicular to $\vec{v}$. Its magnitude can be found from the classic trigonometric relation:

$$|\text{proj}(\vec{u}|\vec{v})| = |\vec{u}|\cos(\theta) \tag{21}$$

and its direction is given by the normalized vector $\vec{v}$. The cosine of the angle θ that the two vectors make with one another can be expressed as:

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \tag{22}$$

where $\vec{u} \cdot \vec{v}$ is the dot product and $|\vec{u}|$ denotes the length of vector $\vec{u}$ calculated as the square root of the dot product of $\vec{u}$ with itself:

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} \tag{23}$$

Put together:

$$\text{proj}(\vec{u}|\vec{v}) = \overbrace{|\vec{u}| \underbrace{\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}}_{\cos(\theta)}}^{\text{magnitude}} \overbrace{\frac{\vec{v}}{|\vec{v}|}}^{\text{direction}} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}||\vec{v}|}\vec{v} \tag{24}$$

The projection is shown in red in Figure 4, and the dotted black line perpendicular to $\vec{v}$ is simply $\vec{u} - \text{proj}(\vec{u}|\vec{v})$.

However, when vectors represent random variables such that each element corresponds to a realization of a random variable in some state, we have to adjust the dot product to accommodate this probabilistic nature. It is done by interchanging it with the expectation dot product. The notion of length $|u|$ changes too, since it is also a dot product:

$$|v| = \left(\mathbb{E}[u^2]\right)^{1/2} \tag{25}$$

Now, substituting all dot products in eq. (24) with expectation dot products we obtain:

$$\text{proj}(u|v) = \frac{\mathbb{E}[uv]}{\mathbb{E}[v^2]}v \tag{26}$$

And hence the desired vector is is:

$$R^{e*} = \text{proj}(1|R^e) = \frac{\mathbb{E}[1R^e]}{\mathbb{E}[(R^e)^2]}R^e \tag{27}$$

Since projection of some vector onto a line is the same as projection on any other vector on this line, we can substitute R^e in the above equation with any excess return:

$$R^{e*} = \frac{\mathbb{E}[1R_i^e]}{\mathbb{E}[(R_i^e)^2]}R_i^e \tag{28}$$

Figure 5 zooms in on the center of our two-asset state space and depicts R^* (black asterisk) and R^{e*} (the blue dotted line). By construction R^* is orthogonal to R^{e*}.

The payoff R^{e*} acts as a "means maker" of excess returns: expected value of its product with any excess return is equal to the expectation of that excess return:

$$\mathbb{E}[R^{e*}R_i^e] = \mathbb{E}\left[\frac{\mathbb{E}[R_i^e]}{\mathbb{E}[(R_i^e)^2]}R_i^e R_i^e\right] = \mathbb{E}[R_i^e] \tag{29}$$

3 3D

3.1 Setting

The setting is now:

$$X = \begin{bmatrix} 0.5 & 1 & 1 \\ 1.5 & 0.75 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{30}$$

$$\pi = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \tag{31}$$

$$P = \begin{bmatrix} 0.75 \\ 0.75 \\ 0.5 \end{bmatrix} \tag{32}$$

Returns:

$$R = \begin{bmatrix} 2/3 & 4/3 & 4/3 \\ 2 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} \tag{33}$$

Excess returns:

$$R^e = \begin{bmatrix} 0 & 0 & 0 \\ 4/3 & -1/3 & -4/3 \\ 4/3 & -4/3 & 2/3 \end{bmatrix} \tag{34}$$

AD prices are:

$$Q = (X^{-1}P)^\top = \begin{bmatrix} 0.3 & 0.4 & 0.2 \end{bmatrix} \tag{35}$$

Risk-free rate is:

$$R^f = 1/\mathbb{E}[M] = 10/9 \tag{36}$$

Although we do not have a risk-free asset among the three original ones, we can easily create it synthetically. In a complete market it is possible to reconstruct *any* payoff $\tilde{X}$, risk-free one being no exception, from the set of available payoffs X. To see this, represent any payoff as a linear combination of the original payoffs:

$$\tilde{X} = X'h \tag{37}$$

And find the weights h_i as:

$$h = (X')^{-1}\tilde{X} \tag{38}$$

which is possible whenever X is invertible.

3.2 P=0

What used to be lines is now planes: so, let us draw a plane of excess returns.

As one needs two points to draw a unique line, one needs three points to construct a unique plane. Any plane is determined by its elevation over the origin and its inclination. The inclination is given by the direction of the vector *normal* to the plane: a vector is called normal to a plane if it is perpendicular to all vectors in this plane (points straight up from it). Since *cross product* of two vectors is perpendicular to both of them (and to any other vector in the plane spanned by them), it can be used as the normal. Thus, we need at least two distinct vectors located in the plane to be able to construct it, which is easy with three points. Cross product of two 1×3 vectors $\vec{u}$ and $\vec{v}$ is defined as the determinant of the following matrix:

$$R^e = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u(1) & u(2) & u(3) \\ v(1) & v(2) & v(3) \end{bmatrix} \tag{39}$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors pointing into the direction of state 1, 2 and 3 respectively. Using Sarus' rule, the determinant is calculated as:

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} u(2) & u(3) \\ v(2) & v(3) \end{vmatrix} \vec{i} + \begin{vmatrix} u(1) & u(3) \\ v(1) & v(3) \end{vmatrix} \vec{j} + \begin{vmatrix} u(1) & u(2) \\ v(1) & v(2) \end{vmatrix} \vec{k} \tag{40}$$

Now it is enough to pick one fixed point X_0 with position vector $\vec{x}_0$ in the plane (easy given that we started off with three points) and say that if another generic point X with position vector $\vec{x}$ is also in the plane, we can construct the difference $\vec{x}_0 - \vec{x}$ which will be in the plane as well. Then the equation of the plane is given by the fact that this difference is perpendicular to the vector normal to the plane:

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0 \tag{41}$$

which in the three-dimensional case translates into:

$$\begin{aligned} [n(1), n(2), n(3)] \cdot [x(1) - x_0(1), x(2) - x_0(2), x(3) - x_0(3)] = 0 \\ n(1)\,(x(1) - x_0(1)) + n(2)\,(x(2) - x_0(2)) + n(3)\,(x(3) - x_0(3)) = 0 \end{aligned}$$

Expanding terms:

$$\begin{gathered} n(1)x(1) + n(2)x(2) + n(3)x(3) = d, \\ d = n(1)x_0(1) + n(2)x_0(2) + n(3)x_0(3) \end{gathered} \tag{42}$$

Any $\vec{x}$ with the three elements satisfying equation (42) will be in the plane.

But what about the elevation over the origin? This will be given by the length of vector $\vec{p}$ which is normal to the plane and connects it to the origin. We will skip the derivation of its coordinates.

How shall we map the Euclidean world described in the shaded box onto the stochastic world? First of all, one still needs three points: there goes the completeness of the market. Then, inclination of the plane will be given by the analogue of the normal vector defined as a weighted cross product:

$$n = a \times b \tag{43}$$
$$n(1) = \frac{\pi(2)\pi(3)}{\sqrt{\pi(1)\pi(2)\pi(3)}}(a(2)b(3) - a(3)b(2)) \tag{44}$$
$$n(2) = \frac{\pi(3)\pi(1)}{\sqrt{\pi(1)\pi(2)\pi(3)}}(a(3)b(1) - a(1)b(3)) \tag{45}$$
$$n(3) = \frac{\pi(1)\pi(2)}{\sqrt{\pi(1)\pi(2)\pi(3)}}(a(1)b(2) - a(2)b(1)) \tag{46}$$

where π denotes the probability measure. You can check the definition with the following simple example: let a and b be the unit vectors in the metric defined through the probability vector π:

$$a = \begin{bmatrix} \frac{1}{\sqrt{\pi(1)}} & 0 & 0 \end{bmatrix} \tag{47}$$
$$b = \begin{bmatrix} 0 & \frac{1}{\sqrt{\pi(2)}} & 0 \end{bmatrix} \tag{48}$$

Then $n = a \times b$ is:

$$n = \begin{bmatrix} 0 & 0 & \frac{\pi(1)\pi(2)}{\sqrt{\pi(1)\pi(2)\pi(3)}}\sqrt{\pi(1)\pi(2)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{\pi(3)}} \end{bmatrix} \tag{49}$$

and its norm $\|v\| = \pi(3)/\sqrt{\pi(3)}^2 = 1$ so it is the unit vector in our inner product space.

Now we have:

$$\mathbb{E}[na] = \mathbb{E}[nb] = 0, \tag{50}$$
$$n = (a \times b) \tag{51}$$

Eq. (50) looks familiar: if instead of u and v we had some excess returns R_1^e and R_2^e, this would be true for the stochastic discount factor! That means that to construct a plane of zero-price returns we only need M and any two such returns. It is only possible to have two excess returns if we started with three payoffs in the first place. M is shown as the thick blue line in Figures 6 and 7. Elevation of the plane is of course equal to the price of payoffs on it: 0 for excess returns, 1 for returns etc.

If we kept payoffs in one of the dimensions constant, all the variation in payoffs with zero price would come from an interplay of the other two: we could thus ask by how much $x(2)$ would change if we increased $x(1)$ such that the price was still zero? In other words, we would like to find the slope of $P = 0$ plane in {state one – state two} direction, which is the analogue of the slope of the lines we drew in 2D. Starting from the plane equation we can again arrive at the familiar relationship:

$$\frac{\partial x(2)}{\partial x(1)} = -\frac{q(1)}{q(2)} = -\frac{\pi(1)}{\pi(2)} \tag{52}$$

In Figures 6 and 7 the angle whose tangent is equal to the above derivative is shown with a gray-shaded sector.

Let us now plot the plane of returns, or payoffs with price $P = 1$. Figures 8 and 9 add this plane to the previous drawing. Since the risk-free rate is positive, the vector $\{1, 1, 1\}$ does not touch or cross the $P = 1$ plane.

To calculate R^{e*} we will make a small detour again since projection of a vector onto a plane is somewhat different from projection onto a line.

In general, projection of $\vec{u}$ onto a k-dimensional plane is calculated as:

$$\text{proj}(u|V, V \in \mathcal{R}^k) = \frac{\vec{u} \cdot \vec{v}_1}{|\vec{v}_1||\vec{v}_1|}\vec{v}_1 + \frac{\vec{u} \cdot \vec{v}_2}{|\vec{v}_2||\vec{v}_2|}\vec{v}_2 + \ldots + \frac{\vec{u} \cdot \vec{v}_k}{|\vec{v}_k||\vec{v}_k|}\vec{v}_k \tag{53}$$

where $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are mutually orthogonal vectors forming the *basis* of V (put simply, a set of coordinates). It is fairly simple to come up with a basis given some k non-redundant vectors $r_1, r_2, \ldots, r_k$ to start with: one sets the first basis vector to be r_1 and then sequentially calculates perpendicular components of $\vec{r}_2, \vec{r}_3, \ldots$ thus ensuring that the obtained vectors are mutually orthogonal:

$$\begin{aligned}
&v_1 = r_1 \\
&v_2 = r_2 - \text{proj}(r_2|v_1) \\
&v_3 = r_3 - \text{proj}(r_3|\{v_1, v_2\}) \\
&\ldots \\
&v_k = r_k - \text{proj}(r_k|\{v_1, v_2, \ldots, v_{k-1}\})
\end{aligned} \tag{54}$$

When vectors are random, substitute expectation dot products for the dot products in (53) and (54).

We find R^* to be [1.41 1.25 0.38] and R^{e*} to be [-0.27 -0.13 0.66]. Figure 10 adds both of them to the picture. These two payoffs are orthogonal to each other, just as they were in 2D.

3.3 Mean-Variance Frontier

Keeping in mind that returns on the mean-variance frontier (MVF) can be constructed as:

$$R^{mve} = R^* + \lambda R^{e*} \tag{55}$$

for different values of λ, we plot some of these returns, which obviously form a line in our state space (the magenta line in Figure 11). Note that the original three assets are NOT on the MVF, which will also be apparent later.

Let us draw the "usual" MVF in the {standard deviation - expected return} space. By changing λ in (55) we construct a bunch of returns, calculate their standard deviation and expected values, and then plot them together with our friend R^* and the three original R_i in Figure 12.

The frontier is wedge-shaped since we were able to create a risk-free payoff. The returns are indeed located within the MVF bounds and are thus not efficient, as we saw already in Figure 11. Of some interest is R^*: this is the minimum second moment return. Recall that the formula of a circle in our set of axes would be:

$$r^2 = \sigma^2[R_i] + (\mathbb{E}[R_i])^2 \tag{56}$$

where r is the radius. The right-hand side of the above equation is precisely the second non-central moment formula, and minimizing it is the same as finding the only feasible return which lies on the circle with the smallest radius.

Let us return to Figure 11. Each blue dot can be decomposed into a mean-variance efficient return on the magenta line and a noise component orthogonal to the latter:

$$R_i = R^* + \lambda R^{e*} + \varepsilon_i \tag{57}$$

The noise component has mean zero, some non-zero idiosyncratic variance and is orthogonal to the other two payoffs. The noise component is therefore not priced.

4 Figures

Figure 2: **Two-asset economy.** Line of payoffs with zero price. The tangent of the angle which the $P = 0$ line makes with the x-axis is equal to the negative of the ratio of state prices.

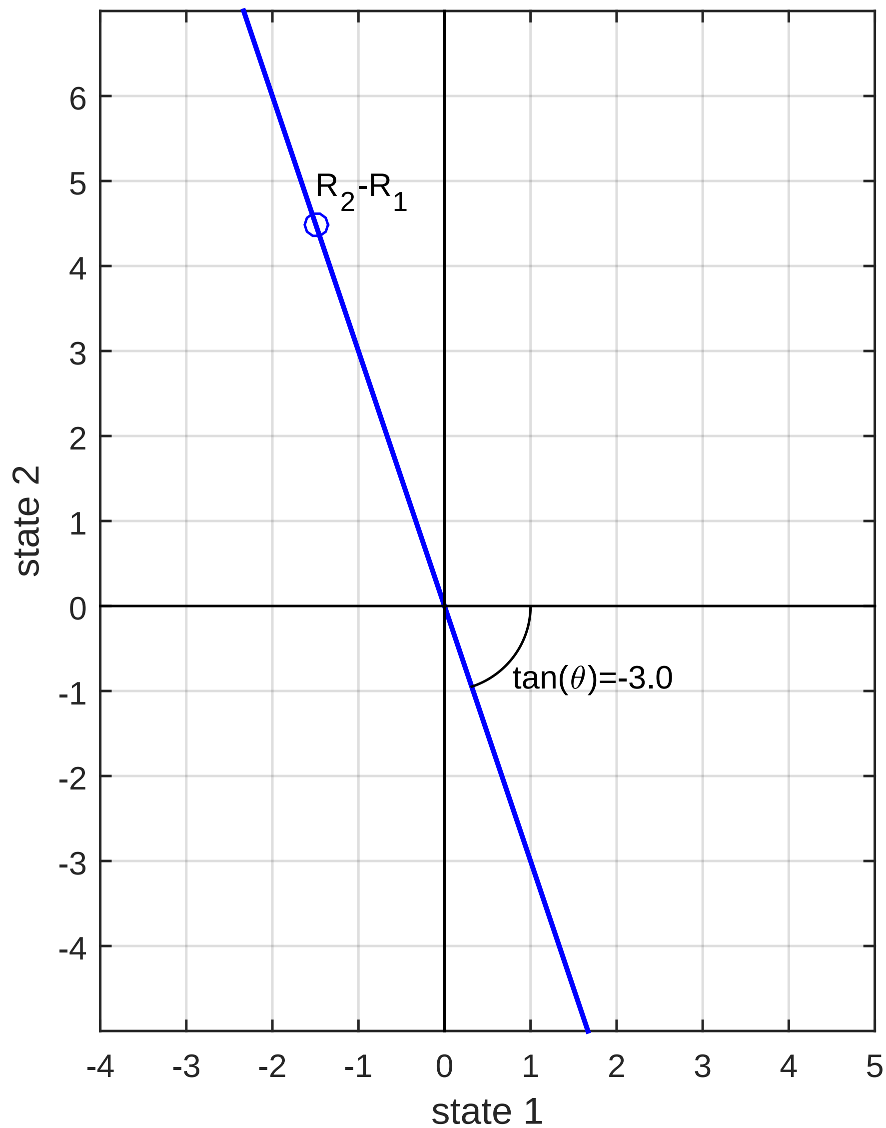


Figure 3: **Two-asset economy.** Lines of payoffs with different prices. All lines have the same slope and intercepts equal to $P/q(2)$. Returns are shown as circles, and the black dot corresponds to the risk-free payoff.

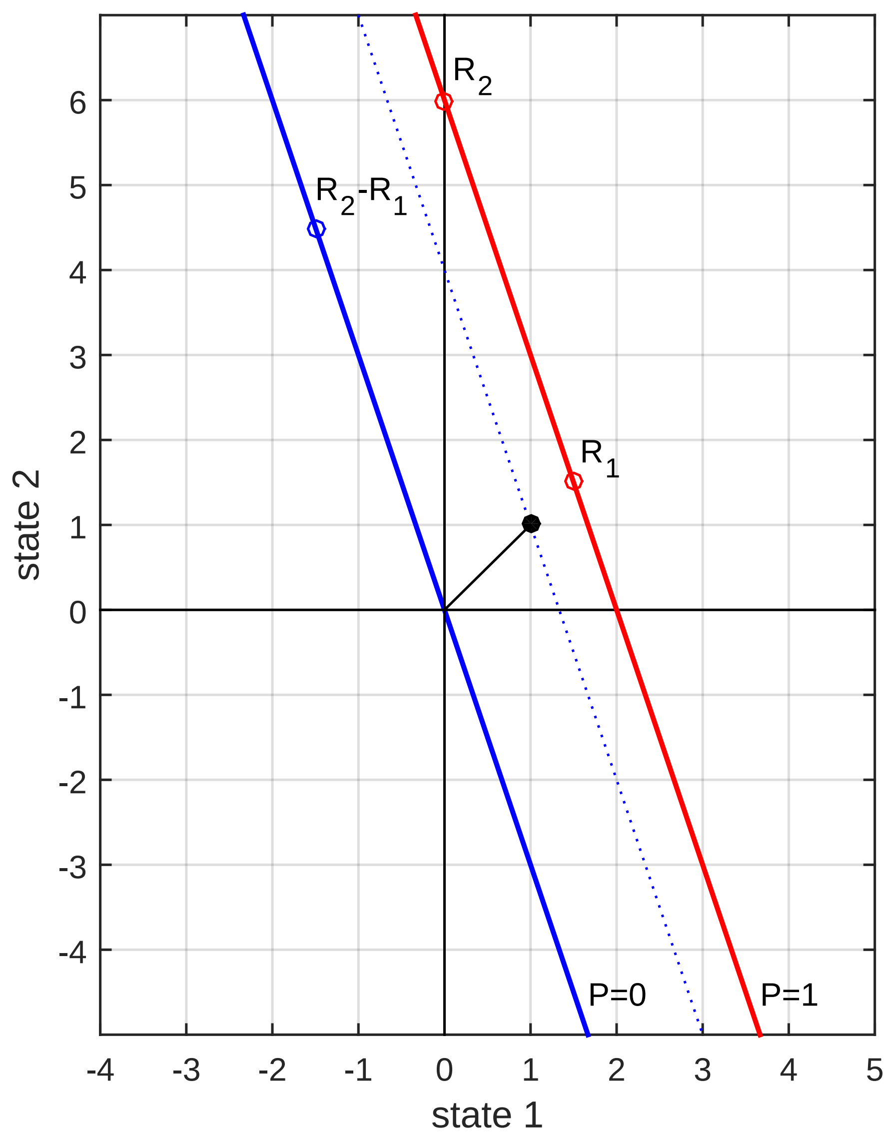


Figure 4: **Example of a projection.**

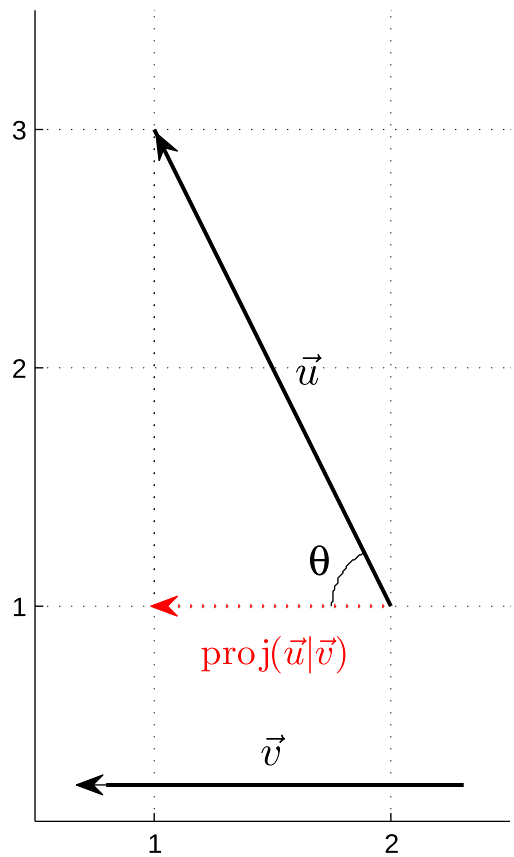


Figure 5: **Two-asset economy.** R^* and R^{e*}. The blue and the red lines host assets with $P = 0$ resp. $P = 1$. Projection of the risk-free payoff onto the former is shown as the dotted blue segment. Dashed lin

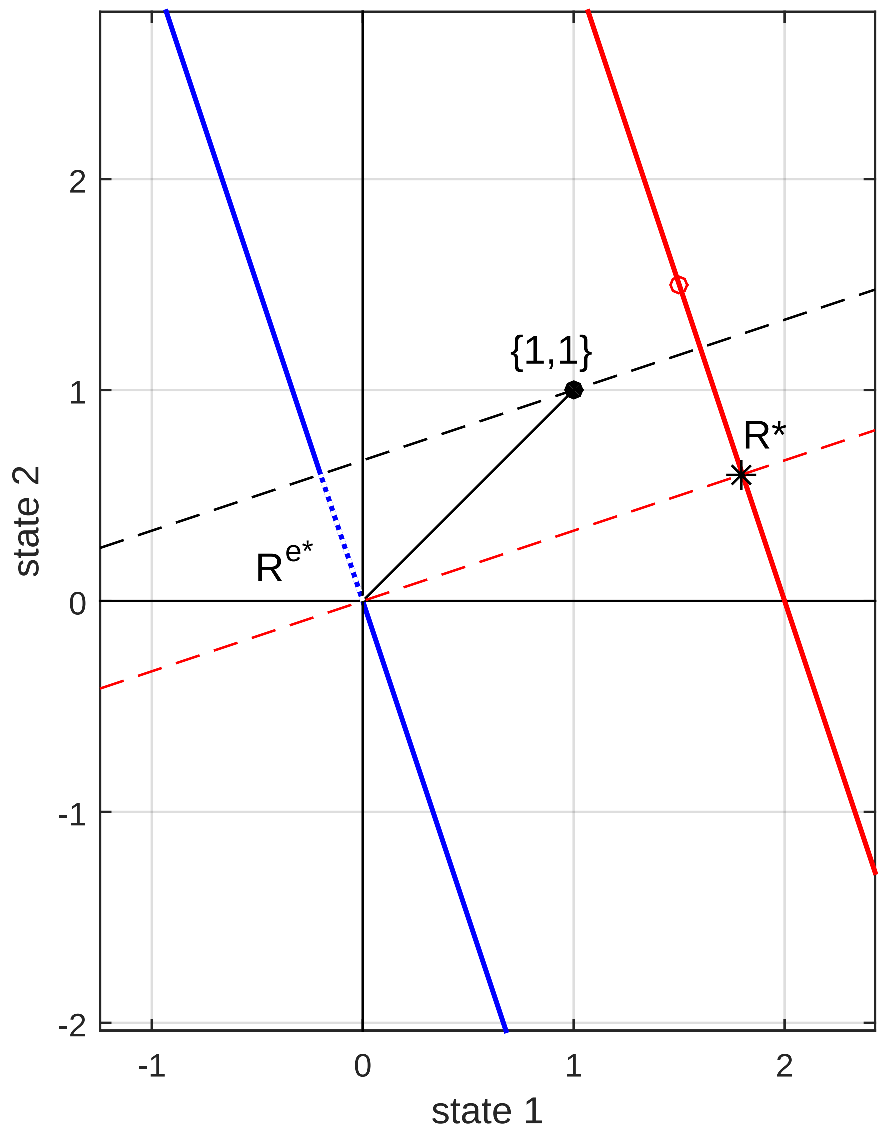


Figure 6: **Three-asset economy.** $P = 0$ plane. The blue line is the normal vector. Red dots are the three original excess returns (one of them being just {0,0,0}). Axes passing through the origin are drawn in dark gray.

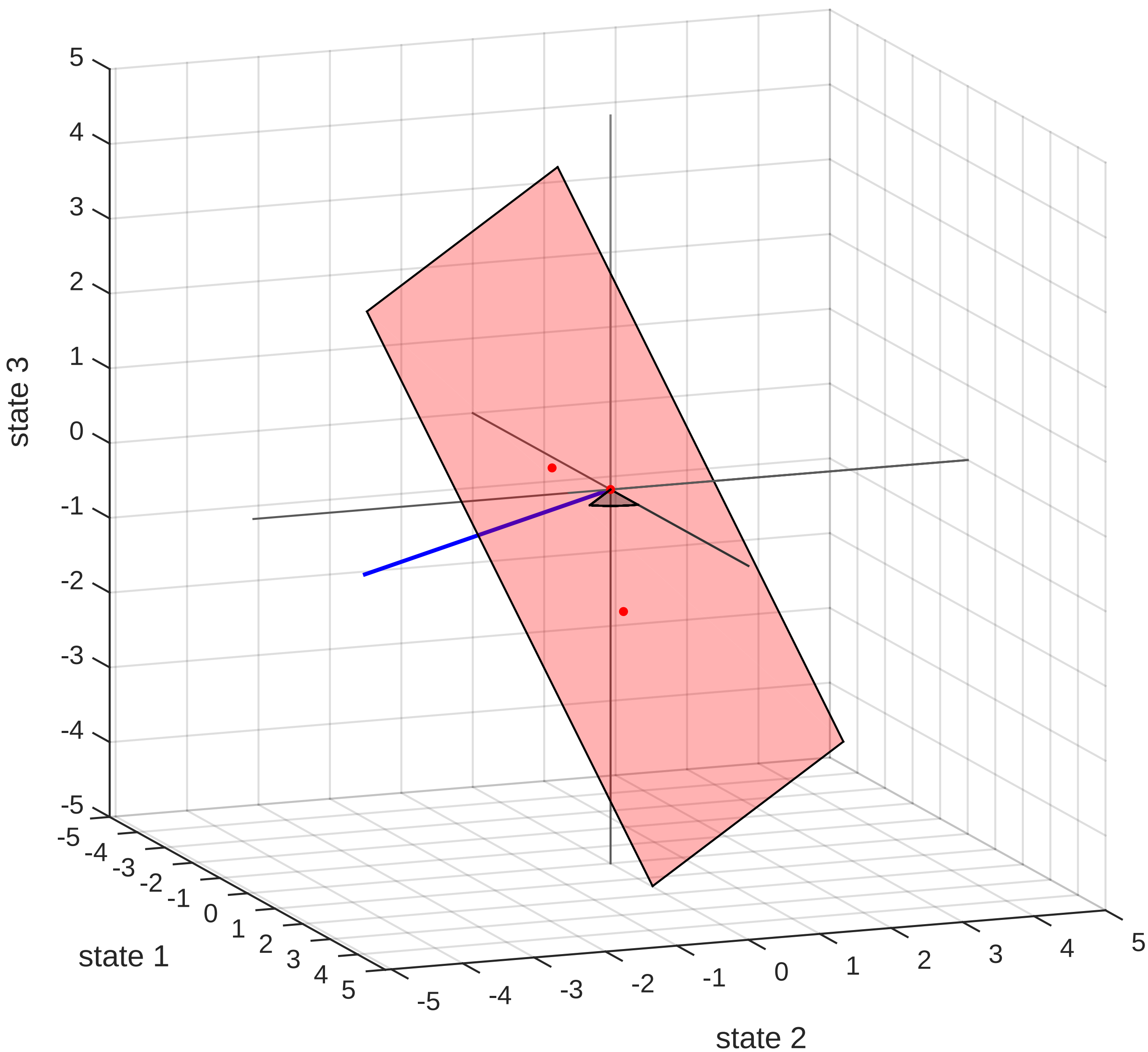


Figure 7: **Three-asset economy.** Figure 6, different view.

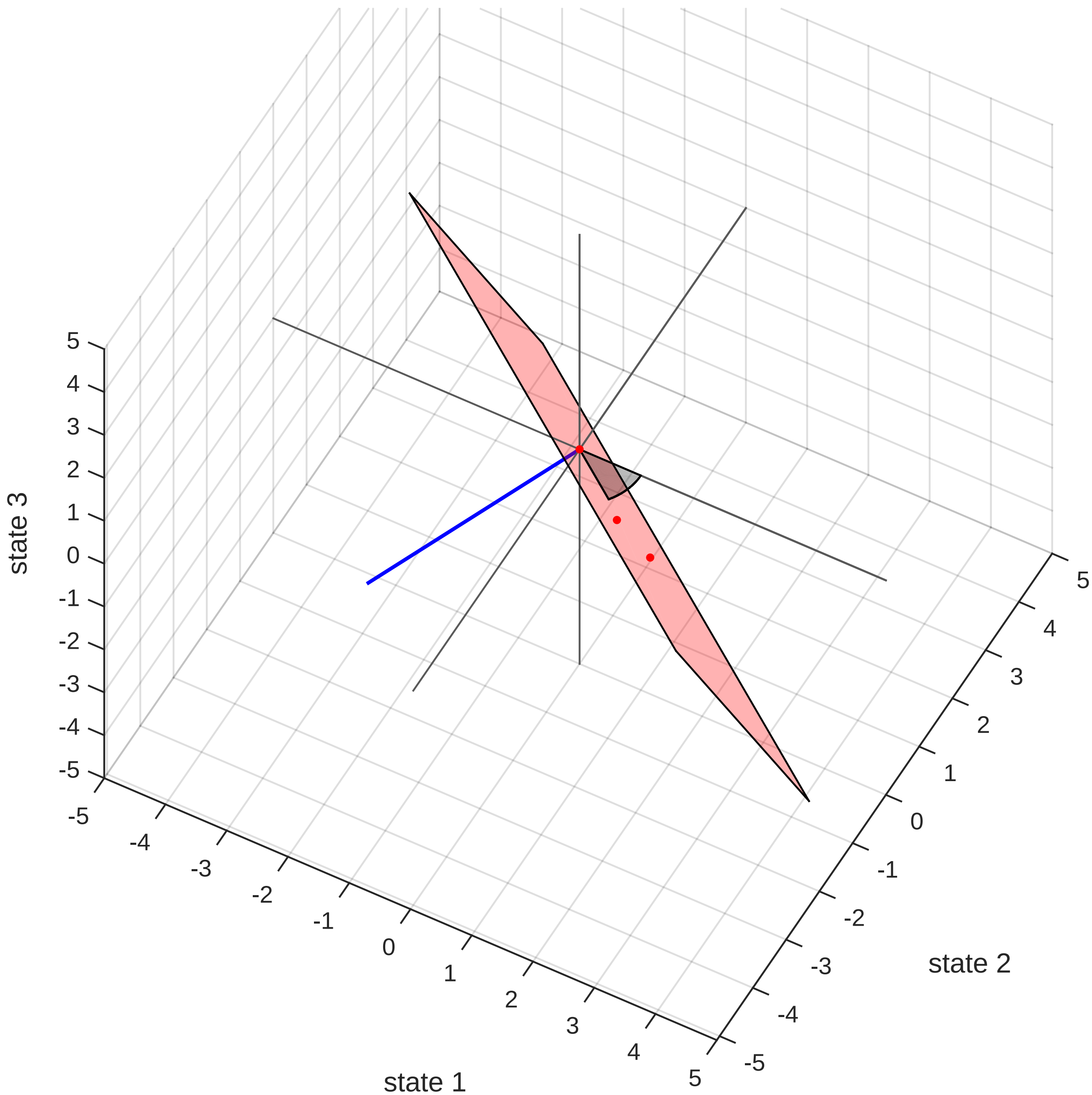


Figure 8: **Three-asset economy.** $P = 0$ plane (red) and $P = 1$ planes (blue). The blue dots are the three original returns. The black dot with a red position vector is the risk-free asset with payoff of 1 in any state: it does not touch or cross the $P = 1$ plane.

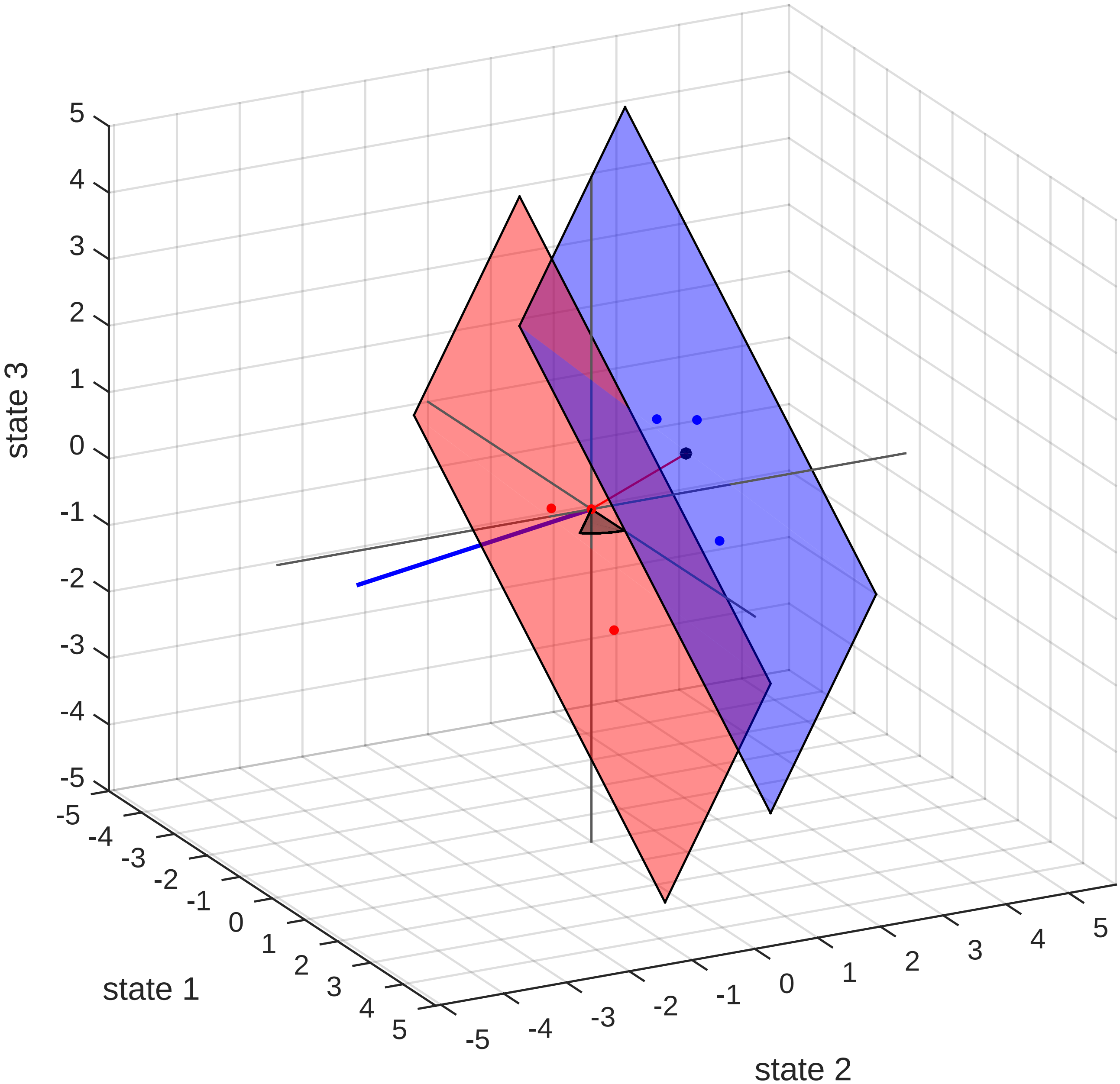


Figure 9: **Three-asset economy.** Figure 8, different view.

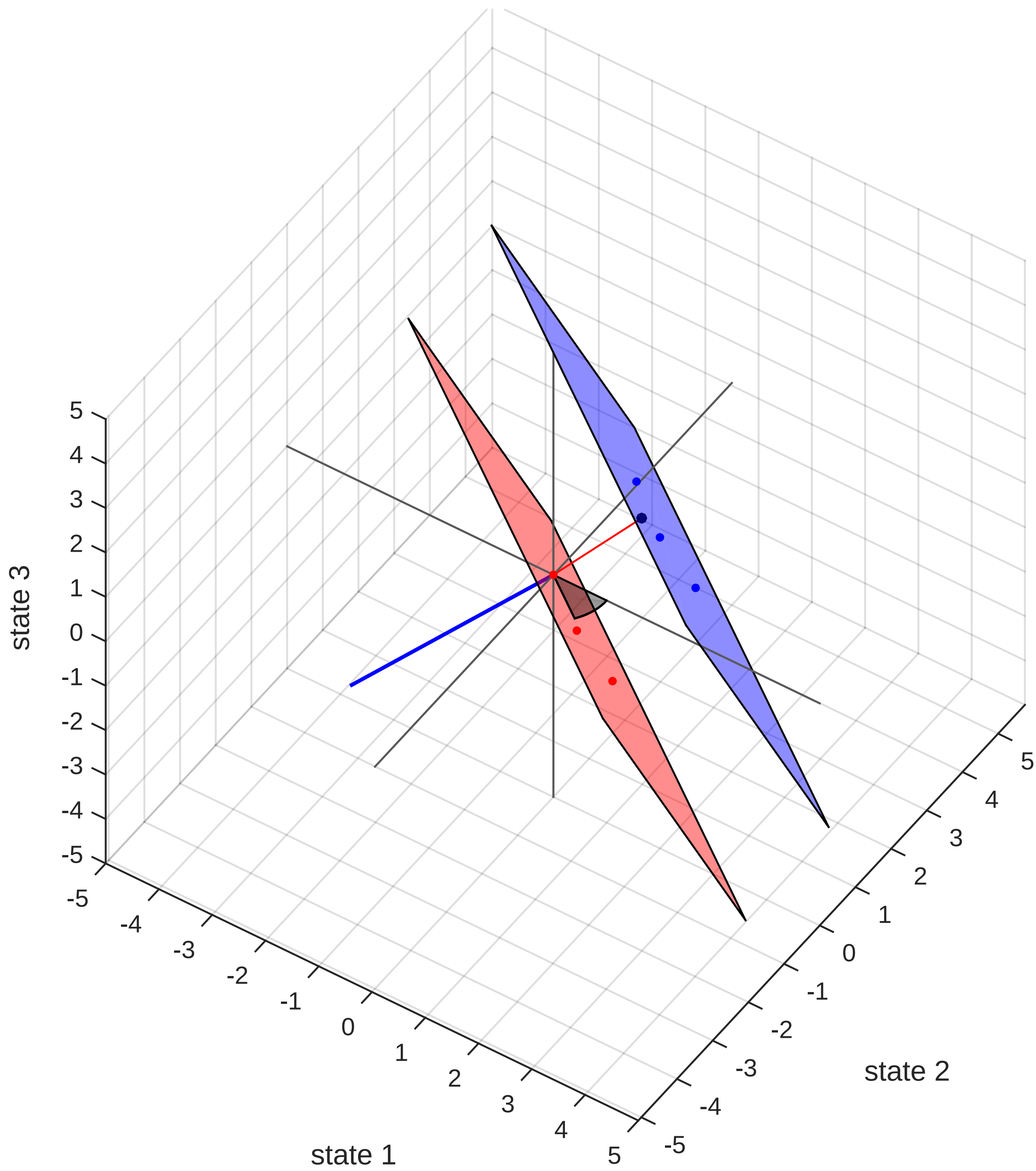


Figure 10: **Three-asset economy.** This figure presents R^{e*}, located on the red surface of excess returns, and R^{*}, located on the surface of returns (not shown here).

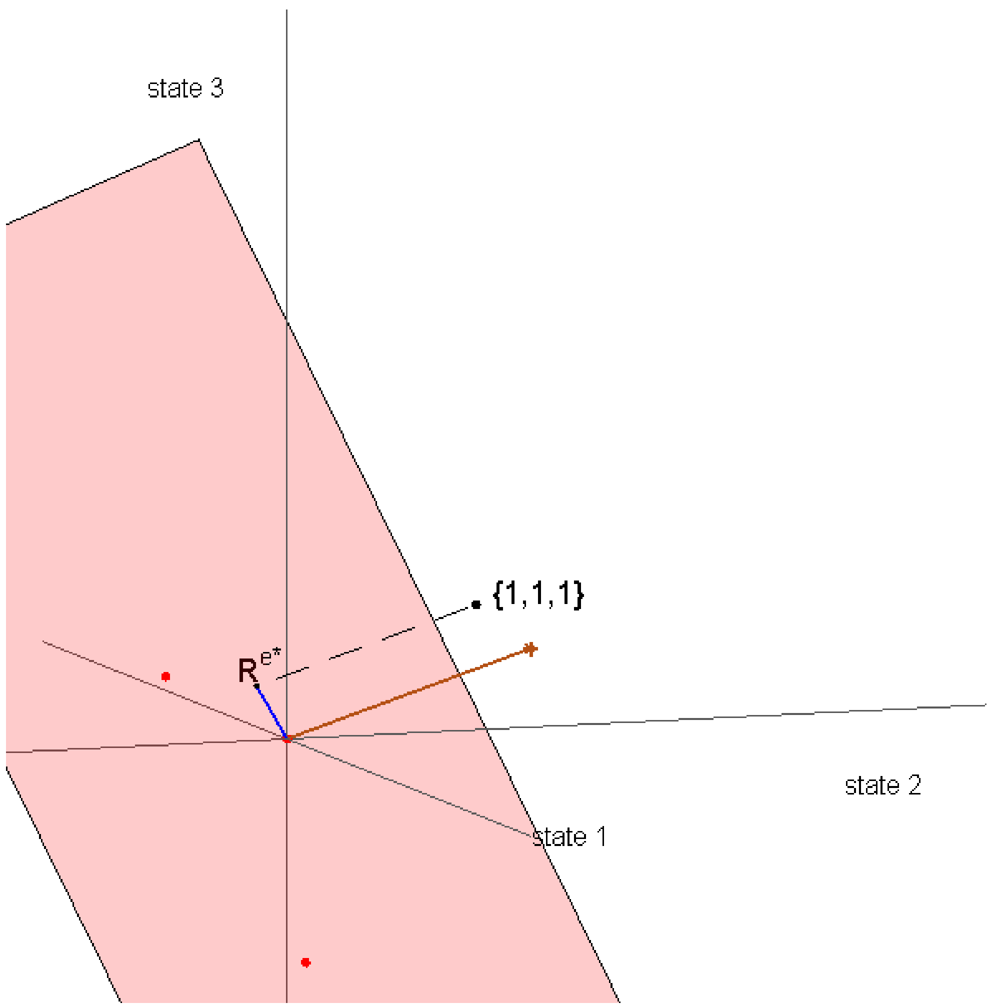


Figure 11: **Three-asset economy.** Mean-variance efficient returns are located on the magenta line. The blue line is R^{e*}, the mahogany line with an asterisk represents R^*, and blue dots on the $P = 1$ blue plane are the original returns.

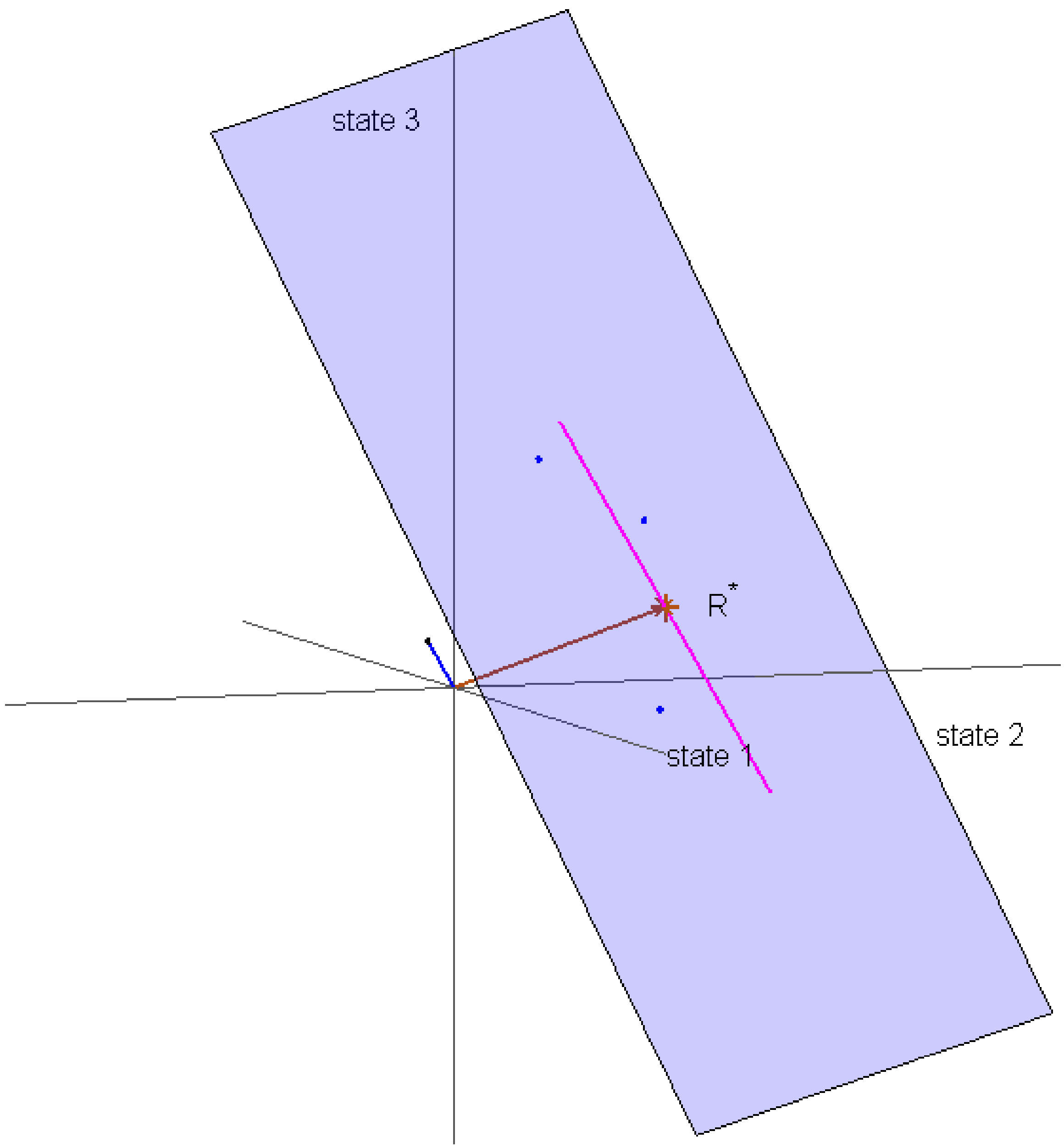


Figure 12: **Three-asset economy.** Mean-variance frontier. Two green dots are two assets on the MVF constructed from (55) with $\lambda_1 = 1.7$ and $\lambda_2 = 2.95$. The red line is the position vector of the R^* payoff. The dashed black line is the arc of the circle with radius equal to the second noncentral moment of R^*. Blue dots represent the three original assets.